SYNTHESIS OF DECOUPLING AND STABILISING CONTROLLER FOR AIRCRAFT LATERAL DYNAMICS

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CERTIFICATE

Certified that this work on SYNTHESIS OF DECOUPLING AND STABILISING CONTROLLERS FOR AIRCRAFT LATERAL DYNAMICS by Flt. Lt. V.K. Parashar has been carried out under my supervision and that this has not been submitted elsewhere for a degree.

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To

Chai ji and Bau ji

whose only asset is our education

and

to

Mata ji, my mother-in-law,

on whom fell the onus of looking after

our children during a critical juncture

of this exercise.

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ABSTRACT

Sideslip and roll responses of an aircraft are coupled calling upon the pilot to use considerable skill to control the aircraft. The requirement that pilot's job be made easier, enabling him, in addition, to achieve better flight-path accuracy, motivates the need to decouple these lateral outputs.

Decoupling is achieved by diagonalising the system transfer function matrix of the lateral set of equations derived, using state feedback. The decoupling problem stated involves the problem of pole-assignment to achieve desired stability characteristic.

The aircraft data made use of is that of an airplane similar to X-15, a high-performance aircraft.

Decoupling controller involving state feedback has been designed for this aircraft. The relative stability of the lateral dynamics of the aircraft has been improved by pole-assignment in order to conform to military handling qualities specifications.

CHAPTER I

INTRODUCTION

In the crowded terminal areas of today's air transportation and for the military requirement of formations of flight during a mission, it would be helpful both to provide a more precise flight-path control and to reduce pilot workload. These two improvements which are often conflicting, can be achieved by providing a decoupled flight control system. The core of the decoupling concept is that for a multivariable system - the aircraft in this case - each control should affect a single output without disturbing others. Pilot workload is reduced since coordination of controls is no longer required. Additionally, flight-path precision can be enhanced since the pilot can focus his attention on a particular output.

It can be appreciated that after decoupling has been achieved, it should be ensured that our system remains stable in its dynamics. Because of the current design trends towards the use of low aspect ratio, sweepback and higher wing loading which form some of the characteristics of high-performance aircrafts, stability of lateral dynamics of the aircraft has become very important. This calls for consideration of motions outside the plane of symmetry, the airplane being free to roll about its x-axis, yaw about its z-axis and translate along its y-axis. The pilot effects these motions by causing

angular deflections of control surfaces. Since the lateral outputs so obtained, broadly specified as the bank angle and the sideslip angle, are often interacting, the pilot is called upon to acquire considerable skill in order to simultaneously manipulate the inputs and successfully control the aircraft. The problem of decoupling and stability must, therefore, be viewed together.

The decoupling problem with its application to aircraft lateral control systems has received some attention and several approaches are possible. One method due to Hall [5,9] and Yore [5,10] uses model following technique. They specified a model that was decoupled and had desirable dynamic characteristics, and then chose a controller so that the original system would follow the model. Though it is a simple approach, it does not always work. In fact, the controller used by Hall minimizes a certain norm but need not make the norm zero. Thus, perfect model following is not assured. For Hall's aircraft if the tangent of a certain reference variable is not zero, the method fails.

Another approach taken by Montgomary and Hatch [7,5], called differential synthesis, is without theory and allows direct calculations of the set of feedback gains and control interconnects that yield arbitrarily selected flying qualities parameters. They specified that the required elements in the output/input transfer function of the aircraft be zero and use

a clever computational method to solve the decoupling problem.

In yet another approach, Cliff and Lutze [5] applied the Geometric Decoupling Theory developed by Wonham and Morse. Following this geometric approach, they first found the most general feedback and control interconnect matrices and then used a quadratic penalty function method to arrive at specified handling qualities. Though this was quite a flexible method, it presented a numerical difficulty in the Gram-Schmidt orthonormalisation procedure which it involved. This difficulty, which necessitated recognition of a certain vector becoming zero during computation, if not overcome, produces erroneous results.

In this work we follow the approach of Falb and Wolovich [2] and Gilbert [1]. The idea is to find feedback and control interconnect matrices such that the output/input transfer function is diagonal and nonsingular. This method, making use of state feedback, also provides a means of achieving a specified pole-placement while simultaneously decoupling the system thus resulting in accomplishment of a stable decoupled system.

Apart from the relative ease of solution of pole-placement, Gilbert's approach, should in principle form the basis for a useful synthesis procedure as his theoretical results were proved by constructive arguments and provided complete structure of the decoupling solution. His methodology was reduced to an

algorithmic form [3,4] and then mechanised by a general purpose computer program which yields all the necessary structural results and synthesis data.

Chapter II of this work derives, ab initio, the lateral directional equations of motion of the aircraft in the state-vector form. Variables of interests and the inputs are defined therein. Chapter III states important results of the decoupling theory, develops the algorithm, explains the steps involved in the computer program and states the decoupling problem. Chapter IV deals with the practical application and Chapter V reports the conclusions stemming from the study.

CHAPTER II

LATERAL DYNAMICS OF AIRCRAFT

The purpose of this chapter is to derive the equations of lateral motion for a rigid airplane in the conventional form. In so doing, several assumptions will have to be made. These assumptions and their implications will be pointed out. The conventional form of the equations will then be transformed into state equations by defining suitable state variables. By specifying the output variables of interest, the output vector equation will then be formed. A brief mention is made about measuring of state variables.

2.1 COORDINATE SYSTEMS AND EXTERNAL FORCES

Consider Fig. 2.1, where the airplane is shown as a body in space. To keep track of its motions, an inertial coordinate system X' Y' Z' is introduced. In most of the problems, earth's motion is sufficiently slow to enable us to consider X' Y' Z' to be earth fixed [21].

The aircraft is a continuum of mass particles dm with position vector $\underline{r}'$ in the X' Y' Z' coordinate system. Each mass particle is subject to the force of gravity. The force per unit volume is,

$$\underline{R} = \rho_A \, \underline{g} \tag{2.1}$$

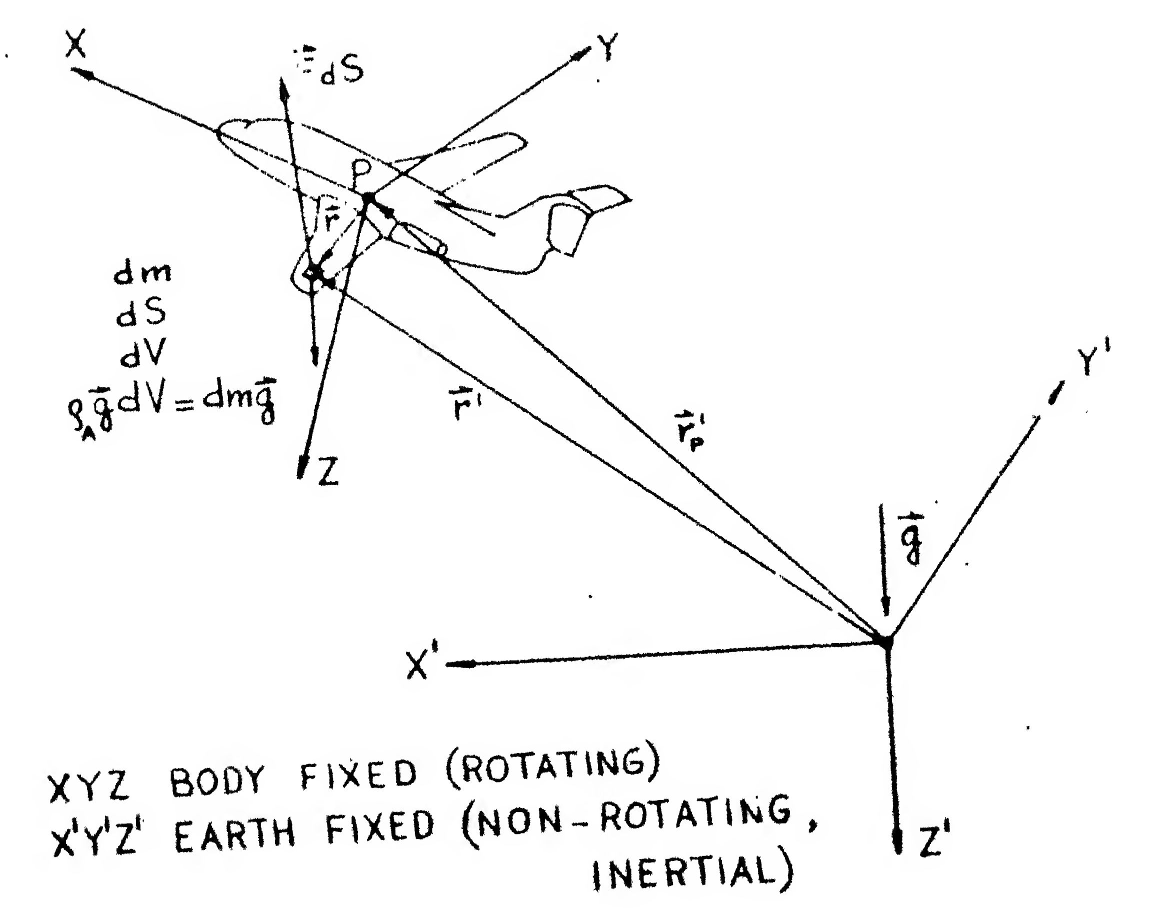


Figure 2.1 Coordinate Systems

where ρ_A is the mass density of the airplane and $\underline{g}$, the acceleration due to gravity. Observe that $\underline{g}$ is always oriented along earth's Z' axis.

Mass particles located at the surface of the aircraft are also subject to an aerodynamic force $\underline{F}$ (including thrust) per unit area. These two external force systems (i.e. $\underline{R}$ and $\underline{F}$) are assumed to be the only ones acting on the airplane.

2.2 EQUATIONS OF MOTION

An aircraft has six degrees of freedom : three translational and three rotational. Conservation of both linear and angular momentum, in terms of Newton's Second Law, yields the following equations of motion;

$$\frac{d}{dt} \int_V \rho_A \frac{d\underline{r}'}{dt} \, dV = \int_V \rho_A \underline{g} \, dV + \int_S \underline{F} \, dS \tag{2.2}$$

(Rate of change of linear Momentum) (Applied forces)

$$\frac{d}{dt} \int_V \underline{r}' \times \rho_A \frac{d\underline{r}'}{dt} dV = \int_V \underline{r}' \times \rho_A \underline{g} \, dV + \int_S \underline{r}' \times \underline{F} \, dS \tag{2.3}$$

(Rate of change of angular Momentum) (Applied moments)

where $\int_V dV$ and $\int_S dS$ represent volume and surface integrals respectively.

The total mass of the airplane is

$$m = \int_V \rho_A \, dV \tag{2.4}$$

We assume that the aircraft mass m is constant, that is,

$$\frac{dm}{dt} = 0 \tag{2.5}$$

This assumption is justified as the change of mass is small over periods of time used in dynamic stability calculations.

It is also assumed that the mass distribution is constant with time. Effects such as fuel slosh are neglected.

Now a new coordinate system X Y Z, having its origin at P, the centre of mass of the airplane is introduced. Point P is kept track of in inertial space by means of the position vector $\underline{r}'_p$. All mass particles are referred to P in X Y Z by means of position vectors $\underline{r}$. X Y Z is called a body-fixed coordinate system. It is rigidly attached to the aircraft and thus moves with it. However, its orientation relative to the airplane remains free to be selected. Clearly, for any mass particle,

$$\underline{r}' = \underline{r}'_p + \underline{r} \tag{2.6}$$

Since P is the centre of mass of the aircraft,

$$\int_V \underline{r} \; \rho_A \, dV = 0 \tag{2.7}$$

This leads to the following definition for $\underline{r}'_p$:

$$\underline{r}'_p = \frac{1}{m} \int_V \rho_A \, \underline{r}' \, dV \tag{2.8}$$

The left hand side of eqn. (2.2) can, therefore, be rewritten as;

$$\frac{d}{dt} \int_V \rho_A \frac{d\underline{r}'}{dt} dV = \frac{d}{dt} \frac{d}{dt} \int_V \rho_A (\underline{r}'_p + \underline{r}) dV = \frac{d}{dt} \frac{d}{dt} m \, \underline{r}'_p$$

$$= m \frac{d}{dt} \frac{d\underline{r}'_p}{dt} = m \frac{d\underline{V}_p}{dt} \tag{2.9}$$

where $\underline{V}_p = \frac{d\underline{r}'_p}{dt}$ is defined as the velocity of the airplane centre of mass.

The right hand side of eqn. (2.2) can be written as

$$\int_V \rho_A \, \underline{g} \, dV + \int_S \underline{F} \, dS = m\underline{g} + \underline{F}_A + \underline{F}_T$$

where $\underline{F}_A$ is the total aerodynamic force and $\underline{F}_T$ is the total thrust force. Eqn. (2.2) would, therefore, become

$$m \frac{d\underline{V}_p}{dt} = m\underline{g} + \underline{F}_A + \underline{F}_T \tag{2.10}$$

Substituting eqn. (2.6) into (2.3) and rearranging,

$$\frac{d}{dt} \int_V \underline{r} \times \frac{d\underline{r}}{dt} \rho_A \, dV = \int_S \underline{r} \times \underline{F} \, dS \tag{2.11}$$

Eqn. (2.11) can be written as

$$\frac{d}{dt} \int_V \underline{r} \times \frac{d\underline{r}}{dt} \rho_A \, dt = \underline{M}_A + \underline{M}_T \tag{2.12}$$

where $\underline{M}_A$ is the total aerodynamic moment vector and $\underline{M}_T$ is the total thrust moment vector.

The volume integral on the left hand side of eqn. (2.12) is a time dependent function. The time dependence can be eliminated by a switch in the coordinate system. By rewriting eqns. (2.10) and (2.12) with respect to system X Y Z instead of X' Y' Z', it turns out that the time-dependence of integration is eliminated. However, system X Y Z is a rotating coordinate system. From classical mechanics [21], it is well known that when a vector $\underline{A}$ is transformed from a fixed to a rotating coordinate system, the following relation holds,

$$\underset{\text{Fixed}(X'Y'Z')}{\frac{d\underline{A}}{dt}} = \underset{\text{Rotating }(X\,Y\,Z)}{\frac{\partial \underline{A}}{\partial t}} + \underline{\omega} \times \underline{A} \tag{2.13}$$

where $\underline{\omega}$ is the angular velocity of system X Y Z with respect to system X' Y' Z' and is obviously identified as the angular velocity of the aircraft since system X Y Z is rigidly attached to it.

Applying eqn. (2.13) to left hand side of (2.10) gives,

$$m\,\frac{d\underline{V}_p}{dt} = m\left(\frac{\partial \underline{V}_p}{\partial t} + \underline{\omega} \times \underline{V}_p\right) = m(\dot{\underline{V}}_p + \underline{\omega} \times \underline{V}_p) \tag{2.14}$$

Therefore,

$$m(\dot{\underline{V}}_p + \underline{\omega} \times \underline{V}_p) = m\underline{g} + \underline{F}_A + \underline{F}_T \tag{2.15}$$

Similarly, the left hand side of eqn. (2.12) can be written as,

$$\frac{d}{dt} \int_V \underline{r} \times \frac{d\underline{r}}{dt} \rho_A \, dV = \int_V \underline{r} \times \frac{d}{dt} \frac{d\underline{r}}{dt} \rho_A \, dV$$

$$= \int_V \underline{r} \times \frac{d}{dt} (\dot{\underline{r}} + \underline{\omega} \times \underline{r}) \rho_A \, dV$$

$$= \int_V \underline{r} \times (\ddot{\underline{r}} + \dot{\underline{\omega}} \times \underline{r} + 2\underline{\omega} \times \dot{\underline{r}} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \rho_A \, dV \tag{2.16}$$

Now, as the aircraft is assumed to be rigid, $\dot{\underline{r}} = \ddot{\underline{r}} = 0$. So, eqn. (2.12) yields,

$$\int_V \underline{r} \times (\dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r})) \rho_A \, dV = \underline{M}_A + \underline{M}_T \tag{2.17}$$

The only time derivative occuring in eqn. (2.17) is $\dot{\underline{\omega}}$, the angular acceleration of the aircraft, which is independent of the volume integration. In other words, the time-dependent integral has indeed been eliminated. Eqns. (2.15) and (2.17) are the vector form equations of motion written in the body-fixed X Y Z system.

The equations of motion (eqns. (2.15) and (2.17)) are now to be expressed in terms of their components along the coordinate directions of the X Y Z system. The coordinate directions are given by the unit vectors i, j and k, in the

usual manner. See Table 2.1 in this connection.

Fig. 2.2 indicates the positive sense and physical meaning of all vector components of Table 2.1. Using eqns. (2.18) in Table 2.1, eqn. (2.15) gets expanded to,

$$
\begin{aligned}
m(\dot{U} - VR + WQ) &= mg_x + F_{A_x} + F_{T_x} \\
m(\dot{V} + UR - WP) &= mg_y + F_{A_y} + F_{T_y} \\
m(\dot{W} - UQ + VP) &= mg_z + F_{A_z} + F_{T_z}
\end{aligned}
\tag{2.19}
$$

To expand eqn. (2.17), rewrite its left hand side using the vector triple product, that is,

$$
\begin{aligned}
&\int_V \underline{r} \times (\underline{\dot{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r}))\, \rho_A\, dV \\
&\quad = \int_V \underline{\dot{\omega}}(\underline{r}.\underline{r})\, \rho_A\, dV - \int_V \underline{r}(\underline{r}.\underline{\dot{\omega}})\, \rho_A\, dV + \\
&\quad \int_V \underline{r} \times \underline{\omega}(\underline{\omega}.\underline{r})\, \rho_A\, dV - \int_V \underline{r} \times \underline{r}(\underline{\omega}.\underline{\omega})\, \rho_A\, dV
\end{aligned}
\tag{2.20}
$$

The first and second term in the right hand side would give,

$$
\begin{aligned}
&\int_V \underline{\dot{\omega}}(\underline{r}.\underline{r})\, \rho_A\, dV - \int_V \underline{r}(\underline{r}.\underline{\dot{\omega}})\, \rho_A\, dV \\
&\quad = (i\dot{P}+j\dot{Q}+k\dot{R}) \int_V (x^2+y^2+z^2)\, \rho_A\, dV - \\
&\quad \int_V (ix+jy+kz)(x\dot{P}+y\dot{Q}+z\dot{R})\, \rho_A\, dV
\end{aligned}
$$

Table 2.1 Definitions of Vector Components of Equations (2.15) and (2.17)

FORCES :

$$\underline{F}_A = iF_{A_x} + jF_{A_y} + kF_{A_z} \qquad (2.18a)$$

for the aerodynamic force components. By a special orientation of the XYZ axes these forces can be shown to be drag, side-force and lift respectively.

$$\underline{F}_T = iF_{T_x} + jF_{T_y} + kF_{T_z} \qquad (2.18b)$$

for the thrust force components.

$$\underline{g} = ig_x + jg_y + kg_z \qquad (2.18c)$$

for the components of gravitational acceleration.

MOMENTS :

$$\underline{M}_A = iL_A + jM_A + kN_A \qquad (2.18d)$$

for the aerodynamic moment components: aerodynamic rolling moment, aerodynamic pitching moment and aerodynamic yawing moment respectively.

$$\underline{M}_T = iL_T + jM_T + kN_T \qquad (2.18e)$$

for the thrust moment components: thrust rolling moment, thrust pitching moment and thrust yawing moment respectively.

VELOCITIES :

$$\underline{\omega} = iP + jQ + kR \qquad (2.18f)$$

for the angular velocity components : roll rate, pitch rate and yaw rate respectively.

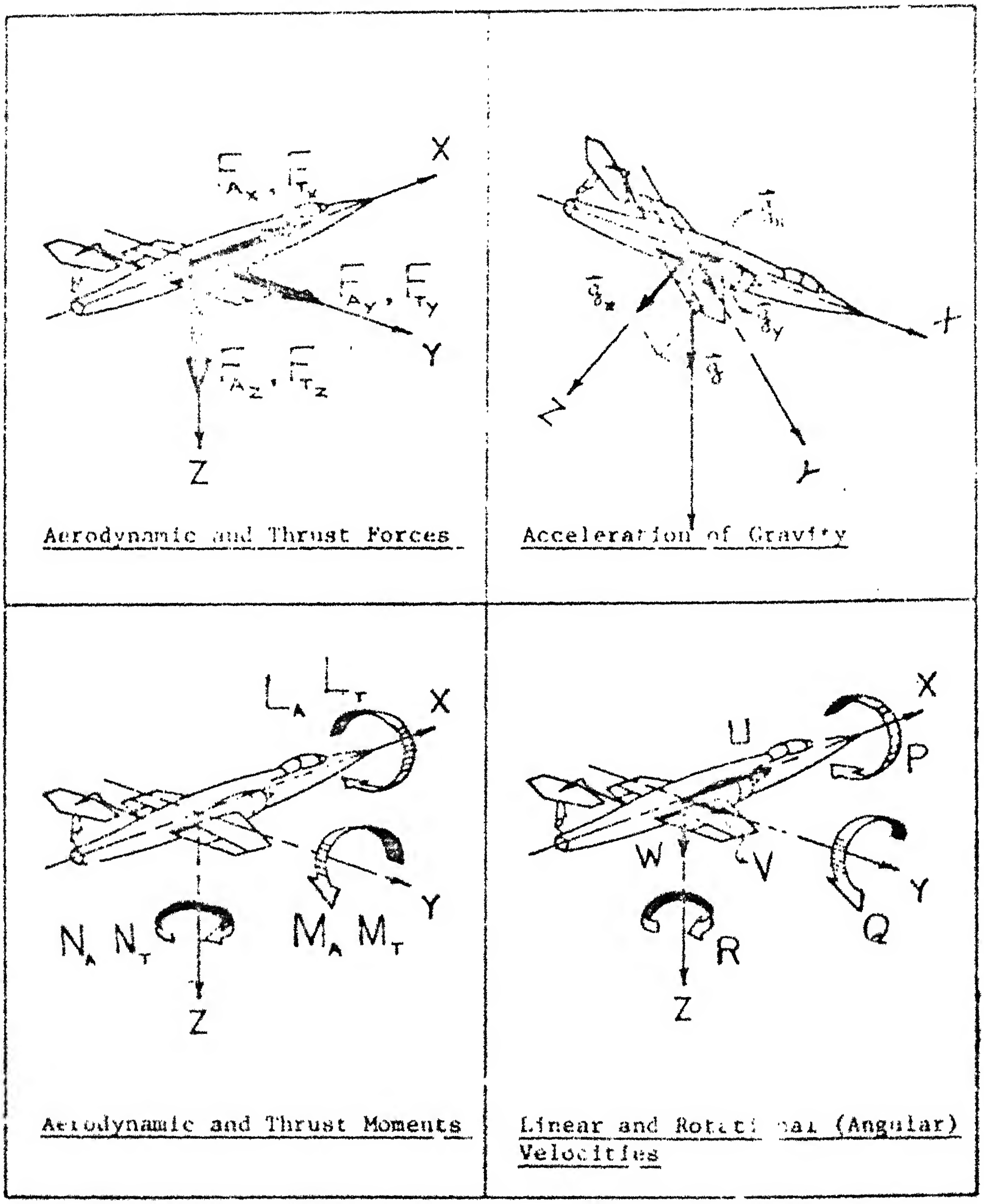
$$\underline{V}_p = iU + jV + kW \qquad (2.18g)$$

for the linear velocity components: forward velocity, side velocity and downward velocity respectively.

DISTANCES :

$$\underline{r} = ix + jy + kz \qquad (2.18h)$$

for the components of the vector $\underline{r}$, which locates mass particles dm inside XYZ.



Note: Positive sense is in the direction of the arrows

Figure 2.2 Definitions of Vector Components in the Equations of Motion

$$= i\{\dot{P}\int_V (y^2+z^2)\ \rho_A\ dV - \dot{Q}\int_V xy\ \rho_A\ dV - \dot{R}\int_V xz\ \rho_A\ dV\} +$$

$$j\{\dot{Q}\int_V (x^2+z^2)\ \rho_A\ dV - \dot{P}\int_V yx\ \rho_A\ dV - \dot{R}\int_V yz\ \rho_A\ dV\} + \tag{2.21}$$

$$k\{\dot{R}\int_V (x^2+y^2)\ \rho_A\ dV - \dot{P}\int_V zx\ \rho_A\ dV - \dot{Q}\int_V zy\ \rho_A\ dV\}$$

The integrals are now recognised as moments and products of inertia. Using their common symbols, the right hand side of eqn. (2.21) becomes

$$= i(\dot{P}I_{xx} - \dot{Q}I_{xy} - \dot{R}I_{xz}) + j(\dot{Q}I_{yy} - \dot{P}I_{xy} - \dot{R}I_{yz}) +$$

$$k(\dot{R}I_{zz} - \dot{P}I_{xz} - \dot{Q}I_{yz}) \tag{2.22}$$

These inertias can be calculated if the size and location of each mass in the aircraft is known. The third term in the right hand side of eqn. (2.20) similarly becomes

$$\int_V \underline{r} \times \underline{\omega}(\underline{\omega}\cdot\underline{r})\ \rho_A\ dV$$

$$= \int_V (ix+jy+kz) \times (iP+jQ+kR)(Px+Qy+Rz)\ \rho_A\ dV$$

$$= i\{I_{xy}\ PR+I_{yz}(R^2-Q^2)-I_{xz}\ PQ+RQ(I_{zz}-I_{yy})\} +$$

$$j\{(I_{xx}-I_{zz})PR+I_{xz}(P^2-R^2)-I_{xy}\ QR+I_{yz}\ PQ\} + \tag{2.23}$$

$$k\{(I_{yy}-I_{xx})PQ+I_{xy}(Q^2-P^2)+I_{xz}\ QR-I_{yz}\ PR\}$$

Most of the airplanes posses a plane of symmetry along their

longitudinal axis. If the X-axis is selected to lie in this plane of symmetry, then $I_{xy} = I_{yz} = 0$. Using this property of aircraft and expressions (2.22) and (2.23), eqn. (2.17) in the component form would yield,

$$I_{xx}\dot{P} - I_{xz}\dot{R} - I_{xz}\, PQ+(I_{zz}-I_{yy})RQ = L_A + L_T \qquad (2.24a)$$

$$I_{yy}\dot{Q} + (I_{xx}-I_{zz})PR + I_{xz}(P^2-R^2) = M_A + M_T \qquad (2.24b)$$

$$I_{zz}\dot{R} - I_{xz}\dot{P}+(I_{yy}-I_{xx})PQ + I_{xz}\, QR = N_A + N_T \qquad (2.24c)$$

Eqns. (2.19) and (2.24) form six differential equations with U,V,W,P,Q and R as six variables.

2.3 ORIENTATION OF THE AIRCRAFT RELATIVE TO THE FIXED COORDINATE SYSTEM X' Y' Z'

Consider system X' Y' Z' translated parallel to itself until its origin coincides with the centre of mass P of the airplane. This translated system named X_1 Y_1 Z_1 is shown in Fig. 2.3.

The relative orientation of X Y Z to X_1 Y_1 Z_1 is described by means of three following consecutive rotations.

i) Coordinate system X_1 Y_1 Z_1 is rotated about the Z_1-axis over an angle ψ , positive as indicated in the figure. This yields coordinate system X_2 Y_2 Z_2. The angle ψ is referred to as the heading (or yaw) angle.

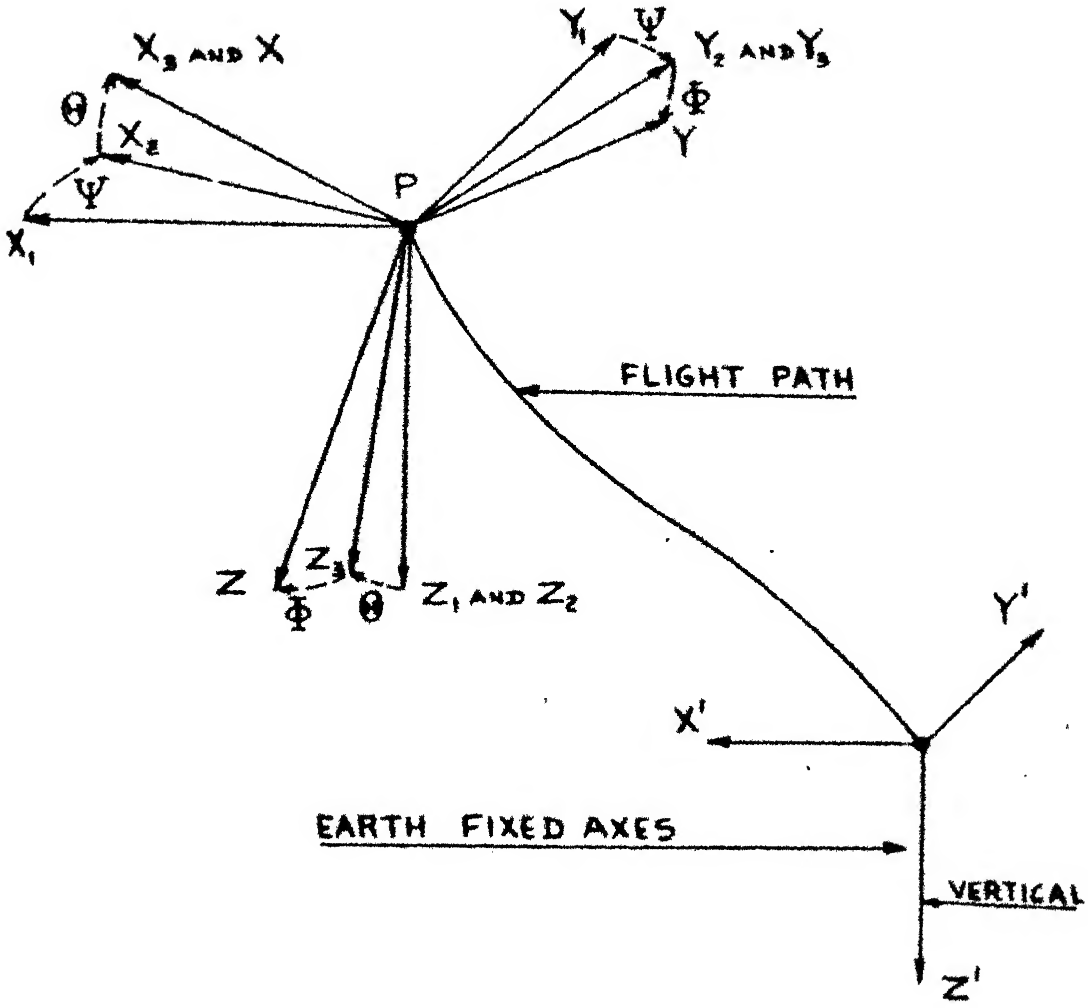


Figure 2.3 Airplane Orientation

ii) Coordinate system $X_2\ Y_2\ Z_2$ is rotated about the Y_2-axis over an angle θ, positive as indicated in figure, yielding coordinate system $X_3\ Y_3\ Z_3$. The angle θ is referred to as the attitude (or pitch) angle.

iii) Coordinate system $X_3\ Y_3\ Z_3$ is rotated about the X_3-axis over an angle φ, positive as indicated in figure. This yields coordinate system X Y Z. The angle φ is referred to as the bank (or roll) angle.

Yaw, pitch and roll angles are called Euler angles.

2.4 COMPONENTS OF GRAVITATIONAL FORCE

Eqn. (2.19) indicates that there are three components of the gravitational force mg. In the body-fixed system X Y Z, these components are mg_x, mg_y and mg_z respectively.

It follows from Fig. 2.1 that,

$$K'g = K_1 g = ig_x + jg_y + kg_z \tag{2.25}$$

where the unit vectors K' and K_1 refer to the X' Y' Z' and $X_1\ Y_1\ Z_1$ systems respectively.

It is desirable to express g_x, g_y and g_z in terms of g and the Euler angles θ and φ (the yaw angle ψ does not affect components of gravity).

From the relationship between the linear and angular velocities of X Y Z and X' Y' Z' systems, it follows that,

$$
\begin{aligned}
g_x &= -g \sin \theta \\
g_y &= g \sin \varphi \cos \theta \\
g_z &= g \cos \varphi \cos \theta
\end{aligned}
\tag{2.26}
$$

These relationships are now substituted in eqn. (2.19) to obtain eqn. (2.27) given below.

2.5 REVIEW OF THE EQUATIONS DERIVED

The force and moment equations, eqns. (2.19) and (2.24) become, respectively,

$$
\begin{aligned}
m(\dot{U} - VR + WQ) &= -mg \sin \theta + F_{A_x} + F_{T_x} \\
m(\dot{V} + UR - WP) &= mg \sin \varphi \cos \theta + F_{A_y} + F_{T_y} \\
m(\dot{W} - UQ + VP) &= mg \cos \varphi \cos \theta + F_{A_z} + F_{T_z}
\end{aligned}
\tag{2.27}
$$

and

$$
\begin{aligned}
I_{xx}\dot{P} - I_{xz}\dot{R} - I_{xz}PQ + (I_{zz}-I_{yy})RQ &= L_A + L_T \\
I_{yy}\dot{Q} + (I_{xx}-I_{zz})PR+I_{xz}(P^2 - R^2) &= M_A + M_T \\
I_{zz}\dot{R} - I_{xz}\dot{P} + (I_{yy}-I_{xx})PQ + I_{xz}QR &= N_A + N_T
\end{aligned}
\tag{2.28}
$$

These equations form six simultaneous nonlinear differential equations of first order in terms of

six variables** U,V,W,P,Q and R.

Solutions of the above nonlinear equations are useful in certain applications, such as airplane simulators, accident investigation and the analysis of aircraft response to large disturbances. Most airplane motions of practical interest to the aircraft designer are steady state motions (that is, equilibrium flight) and perturbed state motions involving only small perturbations relative to some steady state motion.

It is the perturbed state motion we are interested in for our type of problem. Perturbed state flight is mathematically described by considering all motion variables as the sum of a steady state quantity (subscript 1) and a perturbation quantity (lower case). Similar substitutions are carried out for the forces and moments. This is the process of linearizing the nonlinear equations describing aircraft motion about its steady state motion.

**Eqns. (2.27) and (2.28) also contain variables θ and φ, which are related to P,Q and R by means of a set of three kinematic equations, which, in fact, should be invoked for complete trajectory of the airplane. But these are being excluded as they would not have a direct bearing on the problem of synthesis we intend taking up ahead.

Eqns. (2.27) and (2.28) would, therefore, yield

$$
\begin{aligned}
& m\{\dot u - (V_1+v)(R_1+r) + (W_1+w)(Q_1+q)\} = \\
& \qquad - mg\sin(\theta_1+\theta) + F_{A_{x_1}} + f_{A_x} + F_{T_{x_1}} + f_{T_x} \\
& m\{\dot v + (U_1+u)(R_1+r) - (W_1+w)(P_1+p)\} = \\
& \qquad mg\sin(\phi_1+\varphi)\cos(\theta_1+\theta)+F_{A_{y_1}}+f_{A_y}+F_{T_{y_1}}+f_{T_y} \\
& m\{\dot w - (U_1+u)(Q_1+q)+(V_1+v)(P_1+p)\} = \\
& \qquad mg\cos(\phi_1+\varphi)\cos(\theta_1+\theta)+F_{A_{z_1}}+F_{A_z}+F_{T_{z_1}}+f_{T_z}
\end{aligned}
\tag{2.29}
$$

$$
\begin{aligned}
& I_{xx}\dot p - I_{xz}\dot r - I_{xz}(P_1+p)(Q_1+q)+(I_{zz}-I_{yy})(R_1+r)(Q_1+q) = \\
& \qquad L_{A_1} + l_A + L_{T_1} + l_T \\
& I_{yy}\dot q+(I_{xx}-I_{zz})(P_1+p)(R_1+r)+I_{xz}\{(P_1+p)^2-(R_1+r)^2\} = \\
& \qquad M_{A_1} + m_A + M_{T_1} + m_T \\
& I_{zz}\dot r - I_{xz}\dot p+(I_{yy}-I_{xx})(P_1+p)(Q_1+q)+I_{xz}(Q_1+q)(R_1+r) = \\
& \qquad N_{A_1} + n_A + N_{T_1} + n_T
\end{aligned}
\tag{2.30}
$$

Define perturbations θ and φ to be such that,

$$
\begin{aligned}
& \cos\theta \simeq \cos\varphi \simeq 1.0 \\
& \sin\theta \simeq \theta \quad \text{and} \quad \sin\varphi \simeq \varphi
\end{aligned}
\tag{2.31}
$$

This restricts the pitch and roll angle perturbations to roughly 15°, which is still sizable from a practical point of view.

Expanding eqns. (2.29) and (2.30) and applying restrictions (2.31), it is found that the eqns. of motion can be

written as in Table 2.2.

Remarks :

i) Parts of the equations in this table underlined with one line are the steady state equations embedded in the perturbed state. Since, the steady state equations are inherently satisfied, they can be eliminated.

ii) A few other terms are underlined with two lines. These are all nonlinear in nature, that is, they contain products of the perturbation variables u,v,w,p,q,r, θ and φ.

At this point, it is assumed that the perturbations are sufficiently small for products to be negligible with respect to the perturbations themselves.

The majority of airplane dynamics problems are concerned with perturbations relative to a steady state for which :

a) no initial side velocity exists : $V_1 = 0$

b) no initial bank angle exists : $\phi_1 = 0$ (2.32)

c) no initial angular velocities exist : $P_1 = Q_1 = R_1 = \dot{\psi}_1 = \dot{\Theta}_1 = \dot{\phi}_1 = 0$

Introducing restrictions (2.32) and neglecting the nonlinear terms mentioned above, equations of perturbed motion simplify to :

$$m(\dot{u} + W_1 q) = -mg \cos \Theta_1 + f_{A_x} + f_{T_x} \tag{2.33a}$$

$$m(\dot{v} + U_1 r - W_1 p) = mg\, \varphi \cos \Theta_1 + f_{A_y} + f_{T_y} \tag{2.33b}$$

$$m(\dot{w} - U_1 q) = -mg\, \theta \sin \Theta_1 + f_{A_z} + f_{T_z} \tag{2.33c}$$

Table 2.2 Perturbation Equations of Motion with the Restrictions $\cos\theta \simeq \cos\phi \simeq 1$; $\sin\phi \simeq \phi$; $\sin\theta \simeq \theta$

(i) $$\underline{m(-V_1R_1 + W_1Q_1)} + m(\dot{u} - V_1 r - R_1 v + W_1 q + Q_1 w) + \underline{\underline{m(-vr + wq)}} = \underline{-mg\sin\Theta_1 + F_{A_{x_1}} + F_{T_{x_1}}} - mg\theta\cos\Theta_1 + f_{A_x} + f_{T_x}$$

(ii) $$\underline{m(U_1R_1 - W_1P_1)} + m(\dot{v} + U_1 r + R_1 u - W_1 p - P_1 w) + \underline{\underline{m(+ur - wp)}} = \underline{mg\sin\Phi_1\cos\Theta_1 + F_{A_{y_1}} + F_{T_{y_1}}} - mg\theta\sin\Phi_1\sin\Theta_1 + mg\phi\cos\Phi_1\cos\Theta_1 + f_{A_y} + f_{T_y} - \underline{\underline{mg\phi\theta\cos\Phi_1\sin\Theta_1}}$$

(iii) $$\underline{m(-U_1Q_1 + V_1P_1)} + m(\dot{w} + U_1 q - Q_1 u + V_1 p + P_1 v) + \underline{\underline{m(-uq + vp)}} = \underline{mg\cos\Phi_1\cos\Theta_1 + F_{A_{z_1}} + F_{T_{z_1}}} - mg\theta\cos\Phi_1\sin\Theta_1 - mg\phi\sin\Phi_1\cos\Theta_1 + f_{A_z} + f_{T_z} + \underline{\underline{mg\phi\theta\sin\Phi_1\sin\Theta_1}}$$

(iv) $$\underline{-I_{xz}P_1Q_1 + (I_{zz} - I_{yy})R_1Q_1} + I_{xx}\dot{p} - I_{xz}\dot{r} - I_{xz}(P_1 q + Q_1 p) + (I_{zz} - I_{yy})(R_1 q + Q_1 r) + \underline{\underline{- I_{xz}pq + (I_{zz} - I_{yy})rq}} = \underline{L_{A_1} + L_{T_1}} + l_A + l_T$$

(v) $$\underline{(I_{xx} - I_{zz})P_1R_1 + I_{xz}(P_1^2 - R_1^2)} + I_{yy}\dot{q} + (I_{xx} - I_{zz})(P_1 r + R_1 p) + I_{xz}(2P_1 p - 2R_1 r) + \underline{\underline{(I_{xx} - I_{zz})pr + I_{xz}(p^2 - r^2)}} = \underline{M_{A_1} + M_{T_1}} + m_A + m_T$$

(vi) $$\underline{(I_{yy} - I_{xx})P_1Q_1 + I_{xz}Q_1R_1} + I_{zz}\dot{r} - I_{xz}\dot{p} + (I_{yy} - I_{xx})(P_1 q + Q_1 p) + I_{xz}(Q_1 r + R_1 q) + \underline{\underline{(I_{yy} - I_{xx})pq + I_{xz}qr}} = \underline{N_{A_1} + N_{T_1}} + n_A + n_T$$

$$I_{xx}\dot{p} - I_{xz}\dot{r} = l_A + l_T \tag{2.34a}$$

$$I_{yy}\dot{q} = m_A + m_T \tag{2.34b}$$

$$I_{zz}\dot{r} - I_{xz}\dot{p} = n_A + n_T \tag{2.34c}$$

These equations form the basis for most studies of airplane dynamics and control.

We shall now proceed to determine expressions for the aerodynamic and thrust forces and moments to be substituted in the above equations. Though, experimental methods (using models) to evaluate these forces and moments have the advantage of achieving rather accurate values, in most preliminary design and parametric design studies theoretical/ empirical methods are employed. In so doing, an aircraft is considered to be broken into a number of components, namely, wing, body (fuselage), horizontal tail and vertical tail. The total force acting on the aircraft in some coordinate direction is then assumed to be the sum of forces acting on its components in the same coordinate direction. In estimating thrust forces and moments which depend on the power-plant characteristics, it is assumed that engine manufacturer's data are available.

2.6 STABILITY AXES

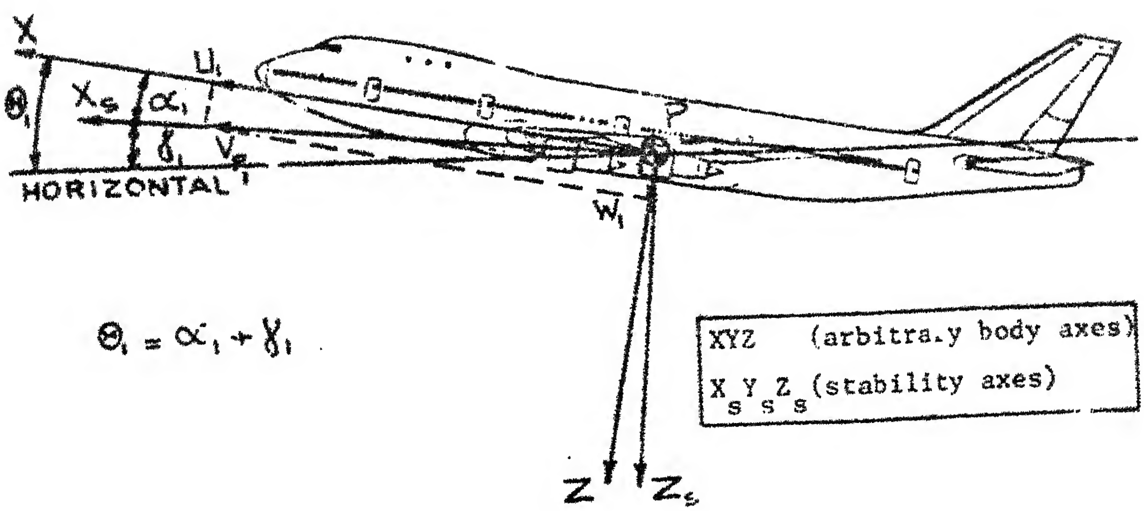
To derive and analyse aerodynamic and thrust forces and moments, it is convenient to use the so-called 'stability axes'

system of coordinates. For this, consider the airplane to be flying a symmetrical steady state straight line flight path. That means $P_1 = Q_1 = R_1 = V_1 = 0$ but U_1 and W_1 are not zero. The angle between the free stream velocity vector V_{P_1} and U_1 is called α_1, the steady state angle of attack. The stability axes X_s Y_s Z_s are obtained from the body axes X Y Z by rotating about $Y = Y_s$ over an angle α_1 until X coincides with V_{P_1} (see Fig. 2.4). The new axes X_s Y_s Z_s are again considered to be rigidly attached to the airplane and thus move with it. It should be evident that the system X_s Y_s Z_s is oriented relative to X Y Z differently for each steady state flight condition. Observe that in the X_s Y_s Z_s system $U_{1_s} = V_{P_1}$ and $W_{1_s} = V_{1_s} = 0$ which leads to further simplification in the equations of motion.

An aircraft in a steady state flight but with $V_1 \neq 0$, is said to be sideslipping. (See Fig. 2.5). Sideslip angle β is defined in Fig. 2.5. The stability axes system in this case is defined in such a way that the X_s axis lies along the projection of the steady state velocity of the aircraft onto the X Z plane.

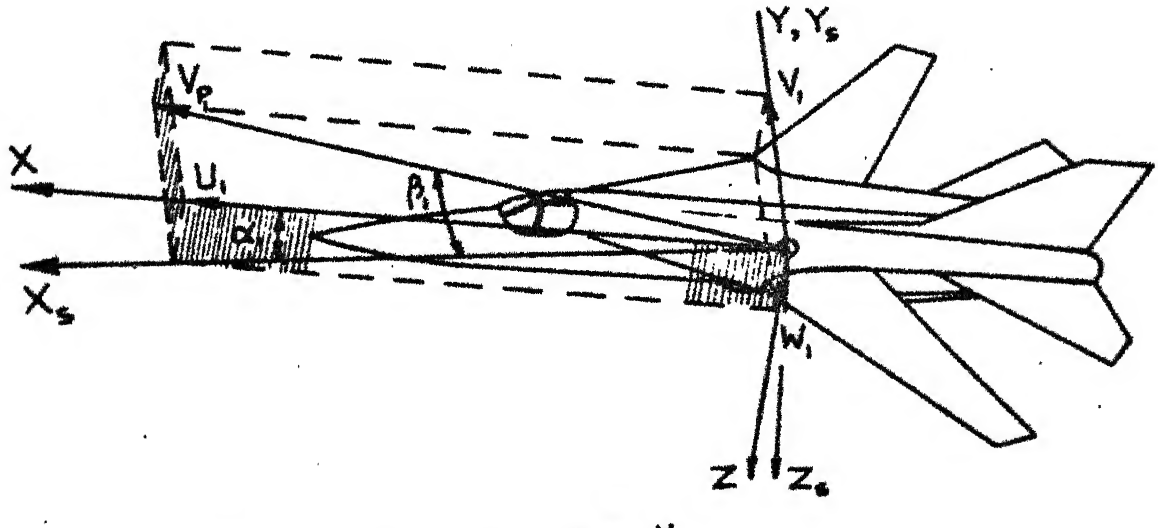
2.7 PERTURBED STATE FORCES AND MOMENTS

Table 2.3 lists the perturbed state motion and control variables and the corresponding forces and moments. In this table δ_A, δ_E and δ_R are, respectively, the aileron, the



$$\Theta_1 = \alpha_1 + \gamma_1$$

Figure 2.4 Definition of Stability Axes



$$\beta_1 = \arc\sin \frac{V_1}{V_{P_1}} \quad \text{or} \quad \beta_1 \cong \frac{V_1}{U_1}$$

Figure 2.5 Definition of Steady State Sideslip Angle

elevator and the rudder deflections which are applied to move the respective control surfaces to cause desired motion of the aircraft. The question, which of the perturbed motion variables u, v, w, p, q, r and their time derivatives affect the perturbed aerodynamic forces and moments is answered by this table. This table in conjunction with Fig. 2.6 gives a systematic analysis of what takes place aerodynamically as perturbations relative to some steady state are introduced. It is clear from this table that it is possible to breakdown the perturbed state forces and moments into independent longitudinal and lateral directional sets. This, actually, is an offshoot of the fact that any inertia coupling is neglected, that is, any longitudinal motions which by definition lie in the plane of symmetry of the aircraft are independent of any lateral directional motions which lie outside this plane of symmetry. Henceforth, our approach/study would be confined to only the lateral motions of the aircraft. We, therefore, branch off with eqns. (2.33b), (2.34a) and (2.34c) that comprise the lateral set and rewrite as :

$$\begin{aligned} m(\dot{v} + U_1 r) &= mg\, \varphi \cos \theta_1 + f_{A_y} + f_{T_y} \\ I_{xx}\dot{p} - I_{xz}\dot{r} &= l_A + l_T \\ I_{zz}\dot{r} - I_{xz}\dot{p} &= n_A + n_T \end{aligned} \tag{2.35}$$

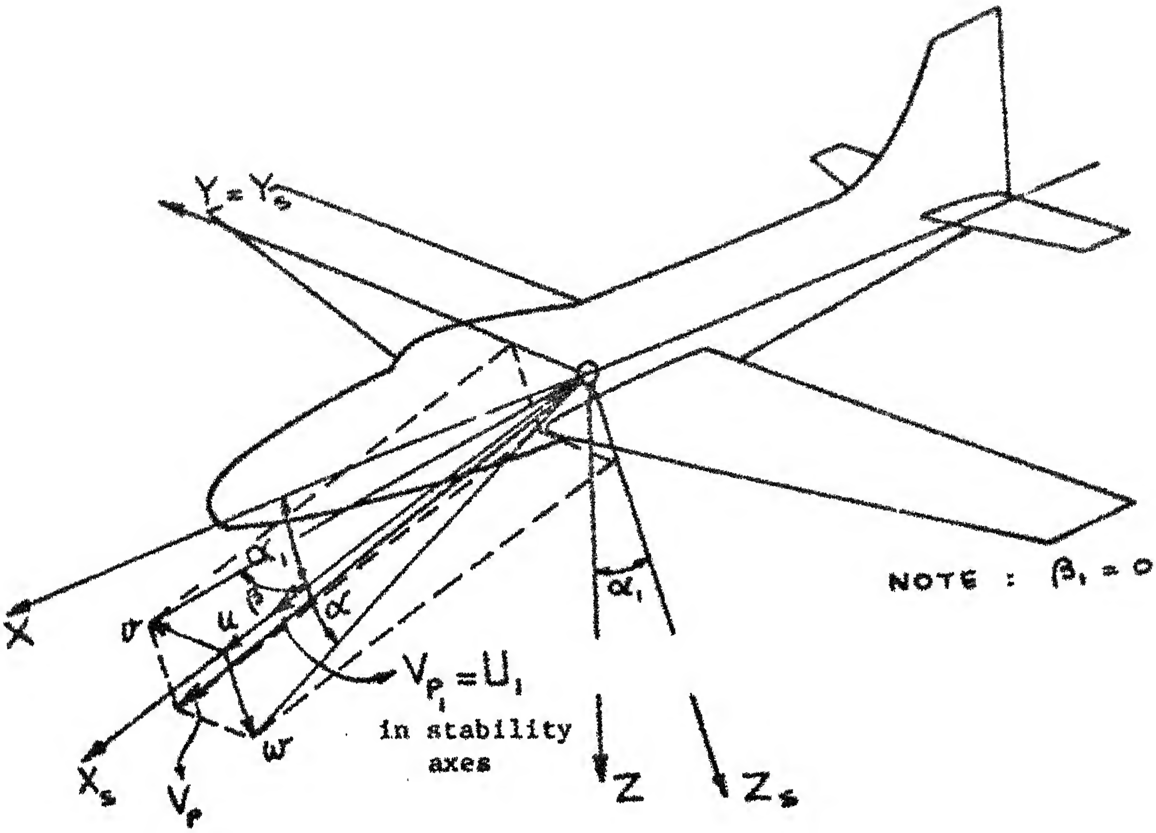
We are, therefore, left with v (or $\beta \simeq \frac{v}{U_1}$), p and r as the only variables of interest. Among time derivatives of these

Table 2.3 Dependence of Perturbed Aerodynamic Forces and Moments on Motion and Control Variables

	DIRECT VARIABLES								DERIVED VARIABLES				CONTROL VARIABLES		
	u	v	w	p	q	r	$\dot{v}$	$\dot{w}$	$\beta=\frac{v}{U_1}$	$\alpha=\frac{w}{U_1}$	$\dot{\beta}=\frac{\dot{v}}{U_1}$	$\dot{\alpha}=\frac{\dot{w}}{U_1}$	δ_A	δ_E	δ_R
f_{A_x}	$\frac{\partial F_{A_x}}{\partial u}$		$\frac{\partial F_{A_x}}{\partial w}$		$\frac{\partial F_{A_x}}{\partial q}$			$\frac{\partial F_{A_x}}{\partial \dot{w}}$		$\frac{\partial F_{A_x}}{\partial \alpha}$		$\frac{\partial F_{A_x}}{\partial \dot{\alpha}}$		$\frac{\partial F_{A_x}}{\partial \delta_E}$	
f_{A_y}		$\frac{\partial F_{A_y}}{\partial v}$		$\frac{\partial F_{A_y}}{\partial p}$		$\frac{\partial F_{A_y}}{\partial r}$	$\frac{\partial F_{A_y}}{\partial \dot{v}}$		$\frac{\partial F_{A_y}}{\partial \beta}$		$\frac{\partial F_{A_y}}{\partial \dot{\beta}}$		$\frac{\partial F_{A_y}}{\partial \delta_A}$		$\frac{\partial F_{A_y}}{\partial \delta_R}$
f_{A_z}	$\frac{\partial F_{A_z}}{\partial u}$		$\frac{\partial F_{A_z}}{\partial w}$		$\frac{\partial F_{A_z}}{\partial q}$			$\frac{\partial F_{A_z}}{\partial \dot{w}}$		$\frac{\partial F_{A_z}}{\partial \alpha}$		$\frac{\partial F_{A_z}}{\partial \dot{\alpha}}$		$\frac{\partial F_{A_z}}{\partial \delta_E}$	
ℓ_A		$\frac{\partial L_A}{\partial v}$		$\frac{\partial L_A}{\partial p}$		$\frac{\partial L_A}{\partial r}$	$\frac{\partial L_A}{\partial \dot{v}}$		$\frac{\partial L_A}{\partial \beta}$		$\frac{\partial L_A}{\partial \dot{\beta}}$		$\frac{\partial L_A}{\partial \delta_A}$		$\frac{\partial L_A}{\partial \delta_R}$
m_A	$\frac{\partial M_A}{\partial u}$		$\frac{\partial M_A}{\partial w}$		$\frac{\partial M_A}{\partial q}$			$\frac{\partial M_A}{\partial \dot{w}}$		$\frac{\partial M_A}{\partial \alpha}$		$\frac{\partial M_A}{\partial \dot{\alpha}}$		$\frac{\partial M_A}{\partial \delta_E}$	
n_A		$\frac{\partial N_A}{\partial v}$		$\frac{\partial N_A}{\partial p}$		$\frac{\partial N_A}{\partial r}$	$\frac{\partial N_A}{\partial \dot{v}}$		$\frac{\partial N_A}{\partial \beta}$		$\frac{\partial N_A}{\partial \dot{\beta}}$		$\frac{\partial N_A}{\partial \delta_A}$		$\frac{\partial N_A}{\partial \delta_R}$

Notes: 1. All perturbations are considered relative to a symmetrical steady state: $V_1 = P_1 = R_1 = 0$

2. Blanks in the table indicate that there is no effect to a first order of approximation



In vector notation:

$\vec{V}_p = \vec{U}_1 + \vec{u} + \vec{v} + \vec{w}$, where $\vec{V}_p$ is the perturbed state total velocity, $\vec{U}_1$ is the steady state total velocity, while $\vec{u},\vec{v},\vec{w}$ are the perturbed velocities

$$\alpha = \arctan \frac{w}{U_1+u} \approx \frac{w}{U_1}$$

$$\beta = \arctan \frac{v}{U_1+u} \approx \frac{v}{U_1}$$

Figure 2.6 Interpretation of Perturbed State Velocities and Angles

motion variables, experience has shown that $\dot{v}$ (or $\dot{\beta}$) is the only one of any practical importance.

In addition are to be considered effects of perturbations in control surface angles. Here it is already assumed that longitudinal control surface perturbations (elevator etc.) do not affect the lateral directional forces and moments. So, we shall only confine to aileron and rudder perturbations δ_A and δ_R respectively.

At this point it is also assumed that the perturbed forces and moments depend only on the instantaneous values of the motion and control variables and not on the time-history of these variables (i.e., quasi-steady assumption).

Finally, it is assumed that it is reasonable to consider only first derivatives of forces and moments with respect to motion and control variables.

With these assumptions, it is now possible to express the perturbed aerodynamic forces and moments as

$$f_{A_y} = \frac{\partial F_{A_Y}}{\partial \beta}\beta + \frac{\partial F_{A_Y}}{\partial \dot{\beta}}\dot{\beta} + \frac{\partial F_{A_Y}}{\partial p}p + \frac{\partial F_{A_Y}}{\partial r}r + \frac{\partial F_{A_Y}}{\partial \delta_R}\delta_R + \frac{\partial F_{A_Y}}{\partial \delta_A}\delta_A$$

$$l_A = \frac{\partial L_A}{\partial \beta}\beta + \frac{\partial L_A}{\partial \dot{\beta}}\dot{\beta} + \frac{\partial L_A}{\partial p}p + \frac{\partial L_A}{\partial r}r + \frac{\partial L_A}{\partial \delta_R}\delta_R + \frac{\partial L_A}{\partial \delta_A}\delta_A$$

$$n_A = \frac{\partial N_A}{\partial \beta}\beta + \frac{\partial N_A}{\partial \dot{\beta}}\dot{\beta} + \frac{\partial N_A}{\partial p}p + \frac{\partial N_A}{\partial r}r + \frac{\partial N_A}{\partial \delta_R}\delta_R + \frac{\partial N_A}{\partial \delta_A}\delta_A \tag{2.36}$$

It is observed that the β, δ_R and δ_A derivatives in (2.36) are dimensionless variables (i.e. radians). It is desirable to have all differentiations take place with respect to dimensionless variables. Division by an appropriate quantity converts these derivatives into their non-dimensional form. Non-dimensional model of eqn. (2.36) would give

$$
\begin{aligned}
f_{A_y} &= \frac{\partial F_{A_y}}{\partial \beta}\beta + \frac{\partial F_{A_y}}{\partial\left(\frac{\dot{\beta} b}{2U_1}\right)}\left(\frac{\dot{\beta} b}{2U_1}\right) + \frac{\partial F_{A_y}}{\partial\left(\frac{pb}{2U_1}\right)}\left(\frac{pb}{2U_1}\right) \\
&\quad + \frac{\partial F_{A_y}}{\partial\left(\frac{rb}{2U_1}\right)}\left(\frac{rb}{2U_1}\right) + \frac{\partial F_{A_y}}{\partial \delta_R}\delta_R + \frac{\partial F_{A_y}}{\partial \delta_A}\delta_A \\
l_A &= \frac{\partial L_A}{\partial \beta}\beta + \frac{\partial L_A}{\partial\left(\frac{\dot{\beta} b}{2U_1}\right)}\left(\frac{\dot{\beta} b}{2U_1}\right) + \frac{\partial L_A}{\partial\left(\frac{rb}{2U_1}\right)}\left(\frac{rb}{2U_1}\right) \\
&\quad + \frac{\partial L_A}{\partial\left(\frac{pb}{2U_1}\right)}\left(\frac{pb}{2U_1}\right) + \frac{\partial L_A}{\partial \delta_R}\delta_R + \frac{\partial L_A}{\partial \delta_A}\delta_A \\
n_A &= \frac{\partial N_A}{\partial \beta}\beta + \frac{\partial N_A}{\partial\left(\frac{\dot{\beta} b}{2U_1}\right)}\left(\frac{\dot{\beta} b}{2U_1}\right) + \frac{\partial N_A}{\partial\left(\frac{pb}{2U_1}\right)}\left(\frac{pb}{2U_1}\right) \\
&\quad + \frac{\partial N_A}{\partial\left(\frac{rb}{2U_1}\right)}\left(\frac{rb}{2U_1}\right) + \frac{\partial N_A}{\partial \delta_R}\delta_R + \frac{\partial N_A}{\partial \delta_A}\delta_A
\end{aligned}
\tag{2.37}
$$

where U_1 is the steady state forward velocity (along X) and b is the span (wing etc.).

Rewriting, in matrix form, eqns. (2.37) in terms of control and stability derivatives which are the quantities expressing the variation of forces and moments on aircraft caused by the disturbance of steady motion and are computable/ measurable, we obtain,

$$\underbrace{\begin{bmatrix} \dfrac{f_{A_Y}}{\bar{q}S} \\[2ex] \dfrac{l_A}{\bar{q}Sb} \\[2ex] \dfrac{n_A}{\bar{q}Sb} \end{bmatrix}}_{\substack{\text{forces} \\ \text{and moments}}} = \begin{bmatrix} C_{y_\beta} & C_{y_{\dot\beta}} & C_{y_p} & C_{y_r} & C_{y_{\delta_A}} & C_{y_{\delta_R}} \\ C_{l_\beta} & C_{l_{\dot\beta}} & C_{l_p} & C_{l_r} & C_{l_{\delta_A}} & C_{l_{\delta_R}} \\ C_{n_\beta} & C_{n_{\dot\beta}} & C_{n_p} & C_{n_r} & C_{n_{\delta_A}} & C_{n_{\delta_R}} \end{bmatrix} \underbrace{\begin{bmatrix} \beta \\ \dfrac{\dot\beta b}{2U_1} \\ \dfrac{pb}{2U_1} \\ \dfrac{rb}{2U_1} \\ \delta_A \\ \delta_R \end{bmatrix}}_{\substack{\text{Non-dimensional} \\ \text{motion variables}}} \tag{2.38}$$

where $\bar{q}$ $(= \frac{1}{2}\rho V_p^2)$ is the dynamic pressure and S is the reference (Wing etc.) surface area.

For a complete discussion of control and stability derivatives, refer [15], pp 4.80 - 4.97.**

Now for the lateral directional perturbed thrust forces and moments, it turns out that only the β variable has a significant effect on the airplane stability. We, therefore, introduce

** See Appendix A for definitions.

$$f_{T_y} = \frac{\partial F_{T_y}}{\partial \beta} \beta$$

$$l_T = \frac{\partial L_T}{\partial \beta} \beta \qquad (2.39)$$

$$n_T = \frac{\partial N_T}{\partial \beta} \beta$$

In terms of β-thrust derivatives, (Refer [15], pp 4.110), eqn. (2.39) is written as,

$$\begin{bmatrix} \dfrac{f_{T_y}}{\bar{q}_1 S} \\ \dfrac{l_T}{\bar{q}_1 Sb} \\ \dfrac{n_T}{\bar{q}_1 Sb} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ C_{n_{T_\beta}} \end{bmatrix} \beta \qquad (2.40)$$

Adding eqns. (2.38) and (2.40) to form the total perturbed forces and moments of the aircraft, we have,

$$\begin{bmatrix} \dfrac{f_{A_y}+f_{T_y}}{\bar{q} S} \\ \dfrac{l_A+l_T}{\bar{q} Sb} \\ \dfrac{n_A+n_T}{\bar{q} Sb} \end{bmatrix} = \begin{bmatrix} C_{y_\beta} & C_{y_p} & C_{y_r} & C_{y_{\delta_A}} & C_{y_{\delta_R}} \\ C_{l_\beta} & C_{l_p} & C_{l_r} & C_{l_{\delta_A}} & C_{l_{\delta_R}} \\ C_{n_\beta}+C_{n_{T_\beta}} & C_{n_p} & C_{n_r} & C_{n_{\delta_A}} & C_{n_{\delta_R}} \end{bmatrix} \begin{bmatrix} \beta \\ \dfrac{\dot{\beta} b}{2U_1} \\ \dfrac{pb}{2U_1} \\ \dfrac{rb}{2U_1} \\ \delta_A \\ \delta_R \end{bmatrix} \qquad (2.41)$$

where, $C_{y_{\dot\beta}}, C_{l_{\dot\beta}}, C_{n_{\dot\beta}} \simeq 0$.

Combining eqn. (2.41) with eqns.of motion (2.35), the lateral directional small perturbation equations in the stability axes system ($W_1 = 0$) obtained are,

$$m(\dot{v}+U_1 r) = mg\,\varphi\,\cos\theta_1 + \bar{q}_1 S(C_{y_{\dot{\beta}}}\beta + C_{y_p}\cdot\frac{pb}{2U_1} + C_{y_r}\cdot\frac{rb}{2U_1} + C_{y_{\delta_A}}\cdot\delta_A + C_{y_{\delta_R}}\cdot\delta_R) \qquad (2.42a)$$

$$I_{xx}\dot{p} - I_{xz}\dot{r} = \bar{q}_1 Sb(C_{l_\beta}\cdot\beta + C_{l_p}\frac{pb}{2U_1} + C_{l_r}\frac{rb}{2U_1} + C_{l_{\delta_A}}\cdot\delta_A + C_{l_{\delta_R}}\cdot\delta_R) \qquad (2.42b)$$

$$I_{zz}\dot{r} - I_{xz}\dot{p} = \bar{q}_1 Sb(C_{n_\beta}\cdot\beta + C_{n_{T_\beta}}\cdot\beta + C_{n_p}\cdot\frac{pb}{2U_1} + C_{n_r}\cdot\frac{rb}{2U_1} + C_{n_{\delta_A}}\cdot\delta_A + C_{n_{\delta_R}}\cdot\delta_R) \qquad (2.42c)$$

2.8 STATE-VECTOR EQUATION OF MOTION

Substituting p (i.e., roll rate) $= \dot{\varphi}$, r (i.e., yaw rate) $= \dot{\psi}$, v (i.e. perturbed side velocity) $= U_1\beta$ and dividing eqn. (2.42a) by m, (2.42b) by I_{xx} and (2.42c) by I_{zz}, we get,

$$\begin{aligned}
&\dot{\beta} - \frac{\bar{q}S}{mU_1} C_{y_\beta}\cdot\beta - \frac{g}{U_1}\varphi + \dot{\psi} = 0 \\
&\ddot{\varphi} - \frac{\bar{q}Sb}{I_{xx}} C_{l_\beta}\cdot\beta - \frac{\bar{q}Sb}{I_{xx}} C_{l_{\delta_A}}\cdot\delta_A - \frac{\bar{q}Sb}{I_{xx}} C_{l_{\delta_R}}\cdot\delta_R = 0 \qquad (2.43)\\
&\ddot{\psi} - \frac{\bar{q}Sb}{I_{zz}} C_{n_\beta}\cdot\beta - \frac{\bar{q}Sb}{I_{zz}} C_{n_{\delta_A}}\cdot\delta_A - \frac{\bar{q}Sb}{I_{zz}} C_{n_{\delta_R}}\cdot\delta_R = 0
\end{aligned}$$

where it is assumed that $I_{xz} = 0$ as both I_{xx} and $I_{zz} >> I_{xz}$ and that the total pitch angle $\theta = 0$ as the flight condition

does not involve any steep ascent/descent of the aircraft. Terms involving the rotary derivatives** and the effect of control authorities on side force*** have been omitted for simplicity. This does not pose any serious restriction on the process of synthesis we intend discussing subsequently.

The control and stability derivatives involved in eqn. (2.43) have been defined in Appendix A.

Eqn. (2.43) finally represents a system of linear differential equations with constant coefficients that describe the perturbed lateral motion of an aircraft.

If $x_1 = \beta$, $x_2 = \varphi$, $x_3 = \dot{\varphi}$, and $x_4 = \dot{\psi}$, then the state of the system may be expressed as a four-component vector $\underline{x}$, or

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \beta \\ \varphi \\ \dot{\varphi} \\ \dot{\psi} \end{bmatrix} \tag{2.44}$$

Therefore,

$$\dot{\underline{x}} = \begin{bmatrix} \dot{\beta} \\ \dot{\varphi} \\ \ddot{\varphi} \\ \ddot{\psi} \end{bmatrix} \tag{2.44a}$$

** C_{l_p}, C_{l_r}, C_{n_p}, C_{n_r}

*** C_{y_p}, C_{y_r}, $C_{y_{\delta_A}}$, $C_{y_{\delta_R}}$

Similarly, the control deflections may be written as a two-component vector $\underline{u}$, or

$$\underline{u} = \begin{bmatrix} \delta_A \\ \\ \delta_R \end{bmatrix} \tag{2.45}$$

The system (eqns. (2.43)) in the state-vector form may be written as

$$\dot{\underline{x}} = A\,\underline{x} + B\,\underline{u} \tag{2.46}$$

where

$$A = \begin{bmatrix} \frac{\bar{q}S}{mU_1} C_{y_\beta} & \frac{g}{U_1} & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \frac{\bar{q}Sb}{I_{xx}} C_{l_\beta} & 0 & 0 & 0 \\ \frac{\bar{q}Sb}{I_{zz}} C_{n_\beta} & 0 & 0 & 0 \end{bmatrix} \tag{2.47}$$

and

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\bar{q}Sb}{I_{xx}} C_{l_{\delta_A}} & \frac{\bar{q}Sb}{I_{xx}} C_{l_{\delta_R}} \\ \frac{\bar{q}Sb}{I_{zz}} C_{n_{\delta_A}} & \frac{\bar{q}Sb}{I_{zz}} C_{n_{\delta_R}} \end{bmatrix} \tag{2.48}$$

The elements of the system matrix A and the control distribution matrix B are constant parameters that depend on flight condition and airplane characteristics.

2.9 OUTPUT EQUATION

Of the four elements of the state-vector defined above, we shall concentrate on φ and β responses of lateral control motion. It will be shown in Chapter III that application of either aileron or rudder by the pilot would result in a coupled response in terms of φ and β. We intend decoupling this response vector for his convenience. So, defining our output vector as,

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \varphi \\ \beta \end{bmatrix}$$

our output description would be

$$\underline{y} = C\,\underline{x}$$

where

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{2.49}$$

Its these matrices A,B and C obtained above that would be input to our algorithm of synthesis procedure and thus get useful results for the solution of decoupling problem.

2.10 MEASUREMENT OF STATE VARIABLES

In the sequel, decoupling is achieved by the application of a control law that involves state feedback. So, the state defined above must be sensed or 'picked up' by some means to be fed to the error detector.

The roll angle φ is measured by a vertical gyro which has its spin axis aligned with the airframe z-axis and is proportional to the voltage from the gyro pick off.

Measurement of sideslip angle β involves measurement of the relative wind direction or the direction of relative motion of air as it passes over the airframe. This is usually accomplished by means of a vane, a probe, dual pressure pick-ups, or some similar device.

The rolling and yawing velocities $\dot{\varphi}$ and $\dot{\psi}$ are measured by the appropriate rate gyros by sensing the torque which is generated by the gyro due to an angular velocity input.

CHAPTER III

DECOUPLING BY STATE FEEDBACK
(with reference to aircraft lateral dynamics)

The purpose of this chapter is to state the decoupling problem and then, by proper state feedback, to find a suitable control law that will not only remove coupling but will also enable the designer to assign system poles to achieve desired stability. Algorithm and computer program and the pioneering work done by Gilbert and Pivnichny to accomplish the above will be explained. It will be shown that aircraft lateral motions are coupled and need to be decoupled. Approach to decoupling problem formulation and solution will be general, i.e., applicable to multivariable nth order system. Reference will be made to two-input two-output fourth order aircraft lateral control system wherever desirable.

3.1 NEED TO DECOUPLE

All aircrafts have to be fitted with a control system that will enable the pilot to trim and maneouvre the aircraft in flight about each of its three axes. On all conventional aircrafts, the controls are 'aerodynamic' ones; the moments required about the centre of gravity of the aircraft being produced as a result of changes in the aerodynamic forces acting on the various parts of the aircraft. In the majority of cases, the changes in the aerodynamic forces are produced by

means of flap-type control surfaces positioned at the extremities of the aircraft, the movement of the pilot's control causing the flap surfaces to move relative to the main structure of the aircraft. There are usually three separate control systems and three sets of surfaces, one set for controlling the aircraft about each of the axes. The pitching control is provided by the elevators, the rolling control by the ailerons and the yawing control by the rudder. The elevators and rudder are usually part of the tail unit and the ailerons are at the wing tips, in order that all these provide the longest possible lever arm about the C.G. of the aircraft.

It is generally desirable that each set of control surfaces should produce a moment only about the corresponding axis. However, in practice, moments are often produced, about the other axes as well. In other words, the effect produced by these is normally coupled. Whereas the action of elevator is, by and large, independent of rudder and aileron, the latter two are interacting. For instance, the yawing moment produced by the aileron deflection and the banking moment caused by rudder action have generally been described as 'adverse' and may, at times, be viewed by the pilot as unpleasant handling qualities. The adverse aileron yaw also tends to reduce the rolling performance of the aircraft, because the yawing produces a sideslip and the dihedral effect produces a rolling moment opposing the desired aileron rolling moment. Similarly,

the rolling moment inevitably produced by rudder control can be termed 'adverse' as it opposes the yaw effect that the rudder may have been put on to produce.

The implication of the above discussion is that the coupling action of aileron and rudder controls not only would reduce their individual desired effects (and hence demanding the pilot to exert more) but would also require the pilot to use considerable skill in order to simultaneously manipulate these two inputs and successfully control the aircraft in lateral direction.

Hence, to the extent that the control of the aircraft in the lateral direction gets easier for the pilot, enabling him to achieve better flight-path accuracy, there exists a need to decouple the φ and β responses.

In fact, many of the control and stability derivatives are an offshoot of coupling effects. Physical explanation of the aerodynamic mechanism of two of these relevant to our problem is given in Appendix B.

Whereas the cross-coupling between the yaw control and the rolling moment and roll control and the yawing moment produced by the application of control surfaces can be avoided by a feedforward mechanism wherein parts of the total (i.e., gross) aileron and rudder deflections are cross-fed to rudder and aileron respectively, for the lateral directional decoupling

to be more effective, the aircraft motions in the roll and yaw modes generated as a result of the application of controls should be decoupled. It is suggested that while the decoupling of moments can be achieved by neutralising the effect of off-diagonal elements of the open loop transfer function matrix, i.e., by so devising the cross-feed as to make the off-diagonal elements zero, the decoupling of cross rolling and yawing motions resulting in offsetting of yawing moment due to rolling motion ($\dot{\varphi}$) and rolling moment due to yawing motion ($\dot{\psi}$) can be effected by diagonalising the system transfer function matrix by state feedback. The algorithm/synthesis scheme made use of in the sequel attempts to achieve the latter.

3.2 THE DECOUPLING PROBLEM AND ITS SOLUTION [1,2,4]

Consider that the time-invariant linear dynamical system $S = \{A,B,C\}$ to be controlled has the form (dotted portion in Fig. 3.1)

$$\dot{\underline{x}} = A\,\underline{x} + B\,\underline{u}$$

$$\underline{y} = C\,\underline{x} \qquad (3.1)$$

where $\underline{u} \in {}^{m}$, $y \in {}^{m}$, $x \in {}^{n}$ and A, B and C are real constant matrices of appropriate size. The transfer function of the system is

$$H(s) = C(sI - A)^{-1}\,B \qquad (3.2)$$

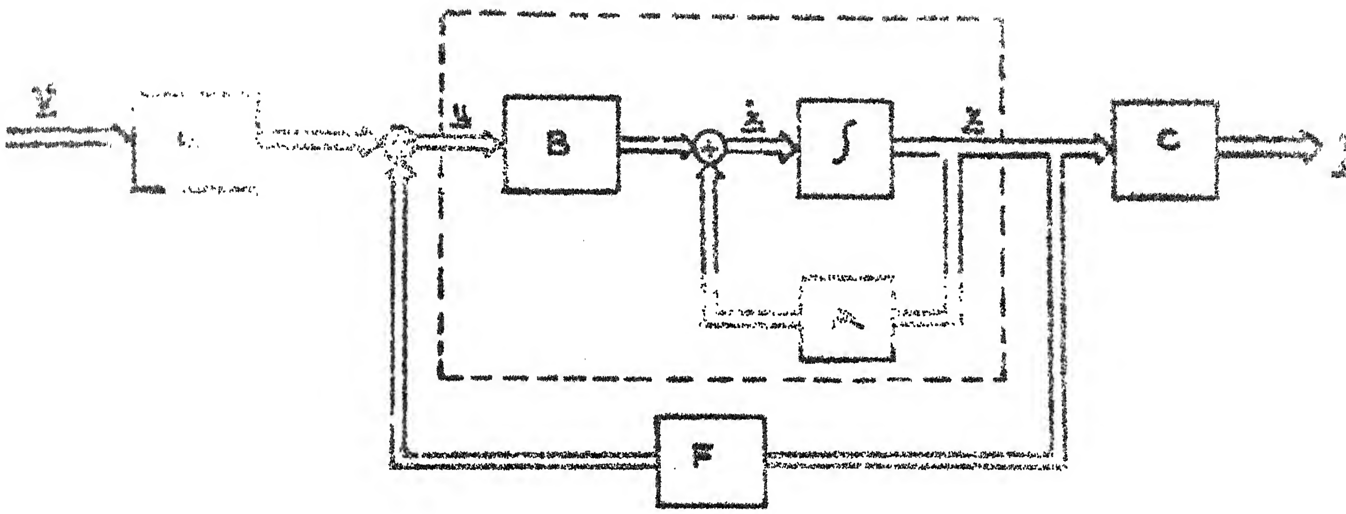


Fig. 3.1 OPEN LOOP & CLOSED LOOP MULTI-INPUT MULTI-OUTPUT SYSTEM

and the system output for zero initial condition is

$$\underline{Y}(s) = H(s)\ \underline{U}(s)$$

or in expanded form as

$$\begin{aligned} y_1(s) &= h_{11}(s)\ u_1(s) + h_{12}(s)\ u_2(s) + \dots + h_{1m}(s)\ u_m(s) \\ y_2(s) &= h_{21}(s)\ u_1(s) + h_{22}(s)\ u_2(s) + \dots + h_{2m}(s)\ u_m(s) \\ &\ \vdots \\ y_m(s) &= h_{m1}(s)\ u_1(s) + h_{m2}(s)\ u_2(s) + \dots + h_{mm}(s)\ u_m(s) \end{aligned} \tag{3.3}$$

These equations are <u>coupled</u>, since each individual input influences all of the outputs.

<u>Definition 1 :</u> A system of the form (3.1) is decoupled (or noninteracting) if its transfer function matrix H(s) is diagonal and nonsingular, i.e.,

$$\begin{aligned} y_1(s) &= h_{11}(s)\ u_1(s) \\ y_2(s) &= h_{22}(s)\ u_2(s) \\ &\ \vdots \\ y_m(s) &= h_{mm}(s)\ u_m(s) \end{aligned} \tag{3.4}$$

We shall now examine some of the principal decoupling results used in this work. Necessary and sufficient conditions were developed by Falb and Wolovitch [2,12] whereas further work leading to a complete synthesis of the decoupling problem was achieved by Gilbert [1].

In what follows, the problem of decoupling will be discussed with respect to the linear state feedback control law of the form

$$\underline{u} = F\,\underline{x} + G\,\underline{v} \tag{3.5}$$

where F is an $m \times n$ real constant matrix, G is an $m \times m$ real constant matrix and $\underline{v}$ is the new m input vector. We will denote the above control law by the pair $\{F, G\}$. G enables the inputs to be rearranged/reordered. The question of interest here is whether such a control law can be used to produce a closed-loop system which is both decoupled and stable.

The desire to decouple raises several practical question, namely, (a) Is decoupling possible? (b) What is the class of control laws which decoupled? (c) What is the class of decoupled closed-loop systems? (d) What is the correspondence between elements of the classes mentioned in (b) and (c)? These four questions constitute the decoupling problem as it is treated in this thesis.

Substituting (3.5) in (3.1), we obtain

$$\begin{aligned} \dot{\underline{x}} &= (A + BF)\,\underline{x} + BG\,\underline{v} \\ \underline{y} &= C\,\underline{x} \end{aligned} \tag{3.6}$$

The transfer function of the closed-loop system $S(F,G) = \{A+BF,\ BG,\ C\}$, thus obtained, is given by

$$H(s,F,G) = C(sI - A - BF)^{-1} BG \tag{3.7}$$

We shall give below the conditions on the open-loop transfer function, $H(s)$, under which the closed-loop transfer function, $H(s,F,G)$, can be made diagonal and nonsingular, so that decoupling is achieved.

Define the non-negative integer d_i as

$d_i \triangleq$ min(the difference of the degree in s of the denominator and the numerator of each entry of the ith row of $H(s)$)-1

and the $1 \times m$ constant row vector $\underline{D}_i$ as

$$\underline{D}_i \triangleq \operatorname*{Lt}_{s \to \infty} s^{d_i+1} \underline{H}_i(s)$$

where $\underline{H}_i(s)$ represents the ith row of $H(s)$.

Let

$$D = \begin{bmatrix} D_1 \\ D_2 \\ \cdot \\ \cdot \\ \cdot \\ D_m \end{bmatrix}$$

<u>Theorem 1 :</u> The system (3.1) may be decoupled by using control law of the form (3.5) if and only if the $m \times m$ matrix D is non-singular.

The above theorem signifies that whether a system can be decoupled or not is a property of its transfer function matrix. The dynamical-equation description comes into play a role only

when the feedback gain matrix F is to be determined. Therefore, the controllability and observability of the system equation are immaterial here.

Expanding (3.2) using Leverrier's algorithm and using the definition of d_i given above, we obtain

$$D_i = C_i A^{d_i} B \neq \underline{0}' \qquad (3.8)$$

where C_i denotes the ith row of C.

Therefore, d_i can also be defined from the open-loop system equations as the smallest integer such that $C_i A^{d_i} B \neq \underline{0}'$. In other words, integer $d_i = j$ where j is the largest integer from $\{1, \ldots, n-1\}$ such that $\underline{C}_i \underline{A}^k B = 0$ for $k = 0, \ldots, j-1$.

If $C_i A^k B \neq \underline{0}'$ for all $k \leq n$, then $d_i = n-1$.

Similarly, we may define $\bar{d}_i$ and $\bar{D}_i$ for the closed-loop system and obtain

$$\bar{D}_i = C_i(A + BF)^{\bar{d}_i} BG \neq \underline{0}' \qquad (3.9)$$

It will be interesting to note that d_i and D_i are not affected by introducing the state feedback, i.e., they are F-invariant.

<u>Definition 2 :</u> An F-invariant of S is any property of S(F,G) which for any fixed G does not depend on F.

<u>Theorem 2 :</u> For any F and any nonsingular G, we have $\bar{d}_i = d_i$ and $\bar{D}_i = D_iG$.

With the above preliminaries, Theorem 1 can be proved establishing its necessary and sufficient conditions as stated below :

Necessity : Suppose that there exists F and G, such that, H(s,F,G) is diagonal and nonsingular. Then,

$$\bar{D} = \begin{bmatrix} \bar{D}_1 \\ \bar{D}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{D}_m \end{bmatrix} = \begin{bmatrix} C_1 A^{d_1} B \\ C_2 A^{d_2} B \\ \cdot \\ \cdot \\ \cdot \\ C_m A^{d_m} B \end{bmatrix} \tag{3.11}$$

is a diagonal constant matrix and nonsingular.

Sufficiency : If the matrix D is nonsingular, then the system can be decoupled.

If we now define another matrix

$$A^* = \begin{bmatrix} C_1 A^{d_1+1} \\ C_2 A^{d_2+1} \\ \cdot \\ \cdot \\ \cdot \\ C_m A^{d_m+1} \end{bmatrix} \tag{3.12}$$

then by confining our choice of F and G to

$$F = D^{-1} A^* \quad \text{and} \quad G = D^{-1} \tag{3.13}$$

the system can be decoupled and the transfer function of the decoupled system is given by

$$H(s,-D^{-1}A^*,D^{-1}) = \begin{bmatrix} \frac{1}{s^{d_1+1}} & & & \bigcirc \\ & \frac{1}{s^{d_2+1}} & & \\ & & \ddots & \\ \bigcirc & & & \frac{1}{s^{d_m+1}} \end{bmatrix}$$

or, equivalently

$$C_i(sI-A-BF)^{-1} BG = \mathrm{diag}\left(\frac{1}{s^{d_i+1}}\right)$$

So, the above theorem specifies a control law which results in m independent subsystems, the ith being equivalent to d_i+1 levels of integration, i.e.,

$$y_i(s) = \frac{1}{s^{d_i+1}} v_i(s) \qquad \text{for } i = 1,2, \ldots, m$$

Such a system which is said to be in Integrator-Decoupled (ID) form, is not satisfactory because all the poles of the decoupled system are at the origin. So, additional feedback will have to be introduced so that while retaining decoupling, it is at the same time, possible to achieve the freedom to alter the pole locations (and hence the dynamic behaviour) of the individual subsystems. To accomplish this, it is convenient to put the system into a canonical form.

3.2.1 Canonically Decoupled Systems

Suppose, we perform a state transformation $\bar{x} = T x$, where T is nonsingular, on the system $S = \{A,B,C\}$ defined by (3.1). The new system, thus obtained would be

$$\bar{S} = \{\bar{A},\bar{B},\bar{C}\} = \{TAT^{-1},\ TB,\ CT^{-1}\} \quad .$$

This motivates the following.

<u>Definition 3 :</u> S and $\bar{S}$ are similar if there exists a nonsingular $n \times n$ matrix T such that $TA = \bar{A}T$, $TB = \bar{B}$ and $C = \bar{C}T$.

In other words, T carries S into $\bar{S}$.

<u>Remark 1 :</u> If S and $\bar{S}$ are similar, their transfer function matrices are equal. Thus $D = \bar{D}$ and $d_i = \bar{d}_i$ for $i = 1,2, \ldots, m$. Therefore, if S is ID, $\bar{S}$ is ID.

<u>Remark 2 :</u> Similar systems are control law equivalent (CLE) implying thereby that a one-to-one correspondence can be established between $\{F,G\}$ and $\{\bar{F},\bar{G}\}$ such that $H(s,F,G) = \bar{H}(s,\bar{F},\bar{G})$.

The main theme of this subsection is to suggest that every ID system is similar to a canonically decoupled (CD) system. This means that if $S = \{A,B,C\}$ can be decoupled, it is possible to find a CD system which is CLE to integral-decoupled S. Since, for a CD system (defined below) the decoupling problem has a particularly simple form, it implies that the treatment of the decoupling problem for S is simplified.

<u>Definition 4 :</u> $S = \{A,B,C\}$ is canonically decoupled (CD) if the following conditions are satisfied :

(i) The matrices A,B and C have the partitioned form :

$$A = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 & 0 & A_1^u \\ 0 & A_2 & 0 & \cdots & 0 & 0 & A_2^u \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_m & 0 & A_m^u \\ A_1^c & A_2^c & A_3^c & \cdots & A_m^c & A_{m+1} & A_{m+1}^u \\ 0 & 0 & 0 & \cdots & 0 & 0 & A_{m+2} \end{bmatrix} \qquad \begin{array}{l} A_i \text{ is } p_i \times p_i, \\ A_i^c \text{ is } p_{m+1} \times p_i, \\ A_i^u \text{ is } p_i \times p_{m+2}, \end{array}$$

$$B = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_m \\ b_1^c & b_2^c & \cdots & b_m^c \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \begin{array}{l} b_i \text{ is } p_i \times 1, \\ b_i^c \text{ is } p_{m+1} \times 1 , \end{array}$$

$$C = \begin{bmatrix} c_1 & 0 & \cdots & 0 & 0 & c_1^u \\ 0 & c_2 & \cdots & 0 & 0 & c_2^u \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c_m & 0 & c_m^u \end{bmatrix} \qquad \begin{array}{l} c_i \text{ is } 1 \times p_i \\ c_i^u \text{ is } 1 \times p_{m+2} , \end{array}$$

where $p_i \geq d_i+1$, $i = 1, \ldots, m$. $\quad (p_i > 0)$

$p_{m+1} = r_{m+1} \geq 0, \qquad p_{m+2} = r_{m+2} \geq 0$.

(ii) For $i = 1, \ldots, m$ the matrices A_i, b_i and c_i have the partitioned form :

$$A_i = \begin{bmatrix} \begin{bmatrix} 0 & I_{d_i} \\ 0 & 0 \end{bmatrix} & 0 \\ \Upsilon_i & \phi_i \end{bmatrix} \qquad \begin{array}{l} \Upsilon_i \text{ is } r_i \times (d_i+1), \\ \phi_i \text{ is } r_i \times r_i, \end{array}$$

$$b_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \gamma_i \\ \beta_i \end{bmatrix}, \; \beta_i = \begin{bmatrix} \beta_{i1} \\ \beta_{i2} \\ \vdots \\ \beta_{ir_i} \end{bmatrix}, \; \gamma_i \text{ are diagonal elements of } D = \Gamma \text{ (say)}.$$

$$c_i = [1 \quad 0 \quad \ldots \quad 0],$$

where $r_i = p_i - 1 - d_i$ and is ≥ 0.

(iii) For $i = 1, \ldots, m$ the systems $S_i = \{A_i, b_i, c_i\}$ are controllable, i.e., the p_i column matrices ... b_i, Ab_i, ..., $A^{p_i-1} b_i$ are linearly independent.

(iv) Let $p = \sum_{i=1}^{m} p_i$ and $\eta = [\eta_1, \eta_2, \ldots, \eta_{m+2}]$ be a partitioned n-row where η_i is a p_i row. Then, if $p_{m+1} \neq 0$ and η is such that $\eta_{p+1}, \ldots, \eta_{p+p_{m+1}}$ are not all zero, then the $1 \times m$ row matrix function $\eta(sI-A)^{-1} B$ has at least two non-zero elements.

Let $\mathcal{L}$ denote the n-dimensional space of n element row matrices. For $i = 1, \ldots, m$, define

$\mathcal{L}_i = \{\eta \mid \eta \in \mathcal{L}_i;\ \eta A^j B_k = 0$ for $k = 1,\ldots,m$, $k \neq i$ and $j=0,\ldots,n-1\}$

where B_i is the ith column of B.

Lemma 1 : Assume $S = \{A,B,C\}$ is ID and controllable. Then for $i = 1, \ldots, m$ the following conditions are satisfied.

(i) $\mathcal{L}_i$ is a row invariant subspace of A, i.e.,

$\eta \in \mathcal{L}_i$ implies $\eta A \in \mathcal{L}_i$;

(ii) $\mathcal{L}_i \cap \mathcal{L}_j = \{0\}$ for $j = 1,\ldots,m$, $j \neq i$;

(iii) $C_i, C_iA, \ldots, C_iA^{d_i}$ are linearly independent elements of $\mathcal{L}_i$.

The above Lemma implies that there exists a linear space $\mathcal{L}_{m+1} \subset \mathcal{L}$ such that the direct sum $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \ldots \mathcal{L}_{m+1} = \mathcal{L}$

Let,

$$p_i \triangleq \dim \mathcal{L}_i,\ i = 1,\ldots, m+1 \quad \text{and } p = \sum_{i=1}^{m} p_i$$

Therefore, $p_{m+1} = n-p$ is uniquely defined by S although $\mathcal{L}_{m+1}$ is not (unless $p_{m+1} = 0$). Also, from part (iii) of the lemma above it follows that $p_i \geq d_i+1$, $i = 1,\ldots, m$.

We can now, state the following lemma.

Lemma 2 : Assume $S = \{A,B,C\}$ is ID, controllable, and $p_{m+1} \neq 0$. Let $\eta \in \mathcal{L}$ and write $\eta \in \sum_{i=1}^{m+1} \xi_i$, where $\xi_i \in \mathcal{L}_i$ for $i = 1,\ldots, m+1$. If $\xi_{m+1} \neq 0$, then there exists at least two integers from the set $\{1,\ldots,m\}$, say q and r, such that $\eta A^j B_q \neq 0$ for at least one $j \in \{0,\ldots,n-1\}$ and $\eta A^j B_r \neq 0$ for at least one $j \in \{0,\ldots, n-1\}$.

Replacing S by $\overline{S}$ in Definition 4 for the CD form and using the results of two lemmas stated above, the following would stand proposed.

<u>Proposition 1 :</u> Assume S is ID and controllable. Then S is similar to a CD system $\overline{S}$, where $\overline{p}_i = p_i$, $i = 1, \ldots, m+1$ and $\overline{p}_{m+2} = 0$.

In the proof of the above proposition, the matrix

$$Q \triangleq \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{m+1} \end{bmatrix}$$

is used, where the rows of the $p_i \times n$ matrix Q_i from a basis for $\mathcal{L}_i$. Thus, the rows of Q (denoted by, say, $q_1^*, q_2^*, \ldots, q_n^*$) form basis for $\mathcal{L}$. Q is used to carry S into $\overline{S}$. To be more specific, we define

$$Q_i = \begin{bmatrix} C_i \\ C_i A \\ \vdots \\ C_i A^{d_i} \\ q^*_{\sigma_i} \\ \vdots \\ q^*_{\rho_i} \end{bmatrix}$$

where $q^*_{\sigma_i}, \ldots, q^*_{\rho_i}$ are any linearly independent rows which extend $C_i, C_iA, \ldots, C_iA^{d_i}$ to form a basis for $\mathcal{L}_i$.

Proposition 1 requires S to be controllable. To make it more general and so applicable to uncontrollable system as well, we need the following lemma.

Lemma 3 : For the nth-order system $S = \{A,B,C\}$, let $n_c = \dim \mathcal{S}$, where $\mathcal{S}$ is the subspace spanned by the mn columns A^jB_k, $j = 0,\ldots, n-1$; $k = 1,\ldots, m$. Then S is similar to $\tilde{S} = \{\tilde{A},\tilde{B},\tilde{C}\}$, where

(i)

$$\tilde{A} = \begin{bmatrix} A^c & A^u \\ 0 & A^U \end{bmatrix} \qquad \begin{array}{l} A^c \text{ is } n_c \times n_c \,, \\ A^u \text{ is } n_c \times (n-n_c), \\ A^U \text{ is } (n-n_c) \times (n-n_c) \end{array}$$

$$\tilde{B} = \begin{bmatrix} B^c \\ 0 \end{bmatrix} \qquad B^c \text{ is } n_c \times m \,,$$

$$\tilde{C} = [C^c \; C^u] \qquad C^c \text{ is } m \times n_c,\; C^u \text{ is } m \times (n-n_c),$$

(ii) $S^c = \{A^c,B^c,C^c\}$ is controllable.

(iii) If S is ID, S^c is ID and $\Gamma^c = \Gamma$ where Γ is diagonal and nonsingular and represents the matrix D.

The procedure required to form a CD representation of an ID system can now be summarised. Let S be an nth order ID system. Apply Lemma 3 obtaining subspace $\mathcal{S}$ and then a matrix T_1 which transforms S into $\tilde{S}$. Apply Proposition 1 replacing S by S^c, which is ID and controllable. The matrix Q

in this case would be $n_c \times n_c$. Define $\hat{p}_{m+2} = n-n_c$ and

$$T_2 = \begin{bmatrix} Q & 0 \\ 0 & I_{\hat{p}_{m+2}} \end{bmatrix}$$

Then direct ecalculation would show that T_2 carries $\bar{S}$ into the CD system $\hat{S}$, where $p_i = \dim \mathcal{-}^c_i$, $i = 1, \ldots, m+1$. Therefore, $T_2\ T_1$ would carry S into $\hat{S}$ resulting in the truth of the following theorem;

<u>Theorem 3 :</u> Every ID system is similar to a CD system.

3.2.2 Principal Results

In pursuance of the fact that the canonical representation of the system is more convenient to work with, the decoupling problem would be solved for the CD system and then by means of Theorem 3, the results can be carried to the original system through back transformations.

<u>Theorem 4 :</u> If S is CD, the control law $\{F,G\}$ decouples S if and only if

$$F = \begin{bmatrix} \theta_1 & 0 & 0 & \cdots & 0 & 0 & \theta^u_1 \\ 0 & \theta_2 & 0 & \cdots & 0 & 0 & \theta^u_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \theta_m & 0 & \theta^u_m \end{bmatrix}$$

where θ_i is $1 \times p_i$ and θ^u_i is $1 \times p_{m+2}$, and

$$G = \operatorname{diag}(\lambda_1, \ldots, \lambda_m), \quad \lambda_i \neq 0, \ i = 1, \ldots, m .$$

Sufficiency of this theorem follows by direct substitution which shows $H(\cdot,F,G)$ is diagonal. In fact, the ith diagonal element of $H(s,F,G)$ is given by

$$h_i(s,F,G) = c_i(Ip_i s - A_i - b_i\Theta_i)^{-1}\, b_i\gamma_i\lambda_i \qquad (3.14)$$

where γ_i are diagonal elements of matrix Γ stated in Lemma 3 part (iii).

In consequence of Theorem 4, we state the next following theorem :

<u>Theorem 5 :</u> Assume S is CD and the control law $\{F,G\}$ has the form indicated in Theorem 4. Then,

(i) the ith diagonal element of $H(s,F,G)$ is

$$h_i(s,F,G) = \frac{\alpha_i(s)\ \gamma_i\ \lambda_i}{\psi_i(s,\sigma_i)}, \quad i = 1,\ldots, m \quad (3.14a)$$

where $\alpha_i(s) = s^{r_i} - \alpha_{i1}s^{r_i-1} - \ldots - \alpha_{ir_i}$ and

$$\psi_i(s,\sigma_i) = s^{p_i} - \sigma_{i1}s^{p_i-1} - \ldots - \sigma_{ip_i},\quad \sigma_i = [\sigma_{ip_i}\ldots\sigma_{i1}];$$

(ii) $\alpha_i(s) = \det(I_{r_i}s - \emptyset_i)$;

(iii) $\Theta_i = (\sigma_i - \pi_i)V_i$, where V_i is a $p_i \times p_i$ nonsingular matrix which depends only on A_i and b_i, and the $1\times p_i$ matrix $\pi_i = [0 \ldots 0\ \alpha_{ir_i} \ldots \alpha_{i1}]$;

(iv) The characteristic equation of the closed-loop system,

$$q(s,F) = \alpha_{m+1}(s)\,\alpha_{m+2}(s) \prod_{i=1}^{m} \psi_i(s,\sigma_i), \quad \text{where}$$

$$\alpha_i(s) = \det(I_{p_i} s - A_i), \; i = m+1,\, m+2 \,.$$

14a

From (3.26), $h_i(s,F,G)$ may be interpreted as the transfer function of the system $S^i(\theta_i, \lambda_i)$ where $S^i = \{A_i, b_i, c_i\}$ is a controllable, single-input, single-output system. So, single variable system theory, summarised in the following lemma, may be applied to prove the above theorem.

<u>Lemma 4 :</u> Assume $S = \{A,b,c\}$ is single input, single output, order n and controllable. Then the transfer function of $S(\theta, \lambda)$ has the form

$$H(s,\theta,\lambda) = \frac{\omega(s)\lambda}{\psi(s,\sigma)}$$

where $\omega(s)$ is a polynomial of degree n-1 or less and $\psi(s,\sigma) = s^n - \sigma_1 s^{n-1} - \dots - \sigma_n$. Let $\sigma = [\sigma_n \dots \sigma_1]$, $\pi = [q_n \dots q_1]$ and define the matrix $K = [k_1 \dots k_n]$, where the columns $k_i = R^{**}_{n-i}\, b$, $i = 1 \dots n$. Then K is nonsingular and $\theta K = \sigma - \pi$, i.e., $\theta = (\sigma - \pi)K^{-1}$.

Except for the form of $\alpha_i(s)$, this lemma directly proves part (i) of the above theorem. From the form of A_i, it is clear that $q_i(s) = \det(I_{p_i} s - A_i) = s^{d_i+1} \cdot \det(I_{r_i} s - \emptyset_i) = s^{d_i+1} \alpha_i(s) = s^{p_i} - \alpha_{i1} s^{p_i-1} - \dots - \alpha_{ir_i} s^{p_i - r_i}$, where we have used the notation of (ii). Letting V_i correspond to K^{-1} of Lemma 4 varifies (iii).

**These are given by Leverrier's expansion [19] and comprise coefficients of the numerator polynomial of $H(s,\theta,\lambda)$.

We now state the final theorem which embraces all the main results stated above;

Theorem 6 : Assume $S = \{A, \dot{B}, C\}$ can be decoupled. If $\{F, G\}$ decouples S, the diagonal elements of $H(\cdot, F, G)$ have the form given in part (i) of Theorem 5 where the integers p_i and r_i and the polynomials $\alpha_i(s)$ are uniquely determined by S, and $\gamma_i = 1$, $i = 1, \ldots, m$. Furthermore, $q(s,F)$ has the form given in part (iv) of Theorem 5, where $\alpha_{m+1}(s)$ and $\alpha_{m+2}(s)$ are polynomials of degree p_{m+1} and p_{m+2} uniquely determined by S. The class of control laws which decouple S can be characterised by $G \in \mathcal{G}$ and $F \in \mathcal{F}$, where $\mathcal{G}$ is an m-dimensional linear span and $\mathcal{F}$ is a $(\sum_{i=1}^{m} p_i + mp_{m+2})$ - dimensional linear manifold. More specifically, there exist matrices $G_i, J_1^i, \ldots, J_{p_i}^i$, $i = 1, \ldots, m$, and an (mp_{m+2}) - dimensional linear space $\mathcal{F}^u$, which are uniquely determined by S, such that

$$G = \sum_{i=1}^{m} \lambda_i G_i \tag{3.15a}$$

$$F = -D^{-1}A^* + \sum_{i=1}^{m} \sum_{k=1}^{p_i} (\sigma_{ik} - \pi_{ik})J_k^i + F^u \tag{3.15b}$$

where $F^u \in \mathcal{F}^u$, $\pi_{ik} = \alpha_{ik}$, $k = 1, \ldots, r_i$, and $\pi_{ik} = 0$, $k = r_i+1, \ldots, p_i$.

The form of $H(\cdot, F, G)$ follows from the CLE property between S and a CD system and Theorem 5. This would yield :

$$G = D^{-1}\Lambda \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m) \tag{3.16a}$$

$$F = -D^{-1}A^{*}+D^{-1}\begin{bmatrix} \theta_1 & 0 & 0 & \ldots & 0 & 0 & \theta_1^u \\ 0 & \theta_2 & 0 & \ldots & 0 & 0 & \theta_2^u \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \theta_m & 0 & \theta_m^u \end{bmatrix} T_2T_1, \tag{3.16b}$$

Results (3.15) are a direct consequences of (3.16), Theorem 4 and Theorem 5. Substitution of (3.16b) into $\det(sI-A-BF)$ leads to the expression for $q(s,F)$.

The questions raised with respect to the decoupling problem get answered now. Part (i) of Theorem 5 tells the designer what closed-loop characteristics are attainable. Once $h_i(s,F,G)$ is specified by fixing σ_i, eqns. (3.15) would give the corresponding control law. If the open-loop system is controllable, $\{F,G\}$ is uniquely determined (F^u would become zero). Part (iv) of Theorem 5 implies that closed-loop characteristic roots are all assignable iff $r_{m+1} = r_{m+2} = 0$ that would make $\alpha_{m+1}(s) = \alpha_{m+2}(s) = 1$. If the polynomials $\alpha_{m+1}(s)$ and $\alpha_{m+2}(s)$ have any root whose real parts are non-negative, it is not possible to design a stable decoupled system.

Theorem 6 establishes all the necessary data (p_i, r_i, α_{ij}, π_{ij}, A^*, D, G_i, J_j^i) needed for the design of decoupled multivariable system. Expressions, formulae and the general approach for obtaining this data have been given in the foregone discussion.

3.2.2 Computer Program : Description and Details

The computer program accepts as input data the matrices A,B and C and produces as output all the data described above for synthesis. The main program, attached as Appendix C to this thesis, consists of 532 statements written in FORTRAN IV language and uses 17 standard SSP [20] subroutines. Names of these subroutines and their operation are annexed with the above appendix. A capability for holding problems of order $n \leq 25$ and input-output dimension $m \leq 10$ has been provided and so requires a large quantity of memory (100,000 bytes approx.). Since the original program was run on IBM System 360, minor changes into it were incorporated in order to make it compatible with the system DEC-1090. These modifications have also been quoted in the annexture.

The program after determining that decoupling is possible ($\det D \neq 0$) proceeds to transform the original system S into a new system $\bar{S}$ which is control law equivalent. Actually three such CLE transformations are made :

$$S \overset{\tau_1}{\Longleftrightarrow} \bar{S} \overset{\tau_2}{\Longleftarrow} \tilde{S} \overset{\tau_3}{\Longleftarrow} \hat{S}$$

Thus, a solution of the decoupling problem for $\hat{S}$, a CD system which has a particularly simple and obvious form, signifies a solution for S. So, this solution when traced back through the same transformations would yield the desired results for synthesis.

The basic steps of the program are outlined in Fig. 3.2.

Step 1 and 2 : Calculation of d_i, D, A^*

Integers $d_1, \ldots, d_m$ are determined by $d_i = j$ where j is the smallest integer such that $C_i A^j B \neq 0$ for $j = 0, \ldots, n-1$ and C_i is the ith row of C. The D and A* follow from (3.11) and (3.12). If det D = 0, decoupling is not possible and the program ends. Else, it proceeds.

Step 3 : Conversion of S to $\bar{S}$ (ID Form)

The transformation between S and $\bar{S}$ is given by

$$\tau_1 : \bar{A} = A - BD^{-1}A^*, \quad \bar{B} = BD^{-1}, \; \bar{C} = C$$

If $$\bar{F} = DF + A^*, \; \bar{G} = DG, \tag{3.17}$$

a direct calculation would show that $H(\cdot, F, G) = \bar{H}(\cdot, \bar{F}, \bar{G})$ proving, thereby, that S and $\bar{S}$ are CIE. A direct calculation also shows that $\bar{d}_i = d_i$ for i = 1, ..., m and

$$\bar{D} = I, \qquad \bar{A}^* = 0 \tag{3.18}$$

If a system satisfies(3.18), it is integrator decoupled since

$$\bar{H}(s, \bar{F}, \bar{G}) = \mathrm{diag}\left(\frac{1}{s^{d_i+1}}\right)$$

Step 4 : Controllable Form of ID System ($\tilde{S}$)

The controllable part of ID system is separated from the uncontrollable part, if any, in this step. The controllability matrix $\bar{H} = [\bar{B}, \bar{A}\,\bar{B}, \bar{A}^2\bar{B}, \ldots, \bar{A}^{n-1}\,\bar{B}]$ is formed and its rank n_c determined. If $n_c = n$, $\bar{S}$ is already controllable and

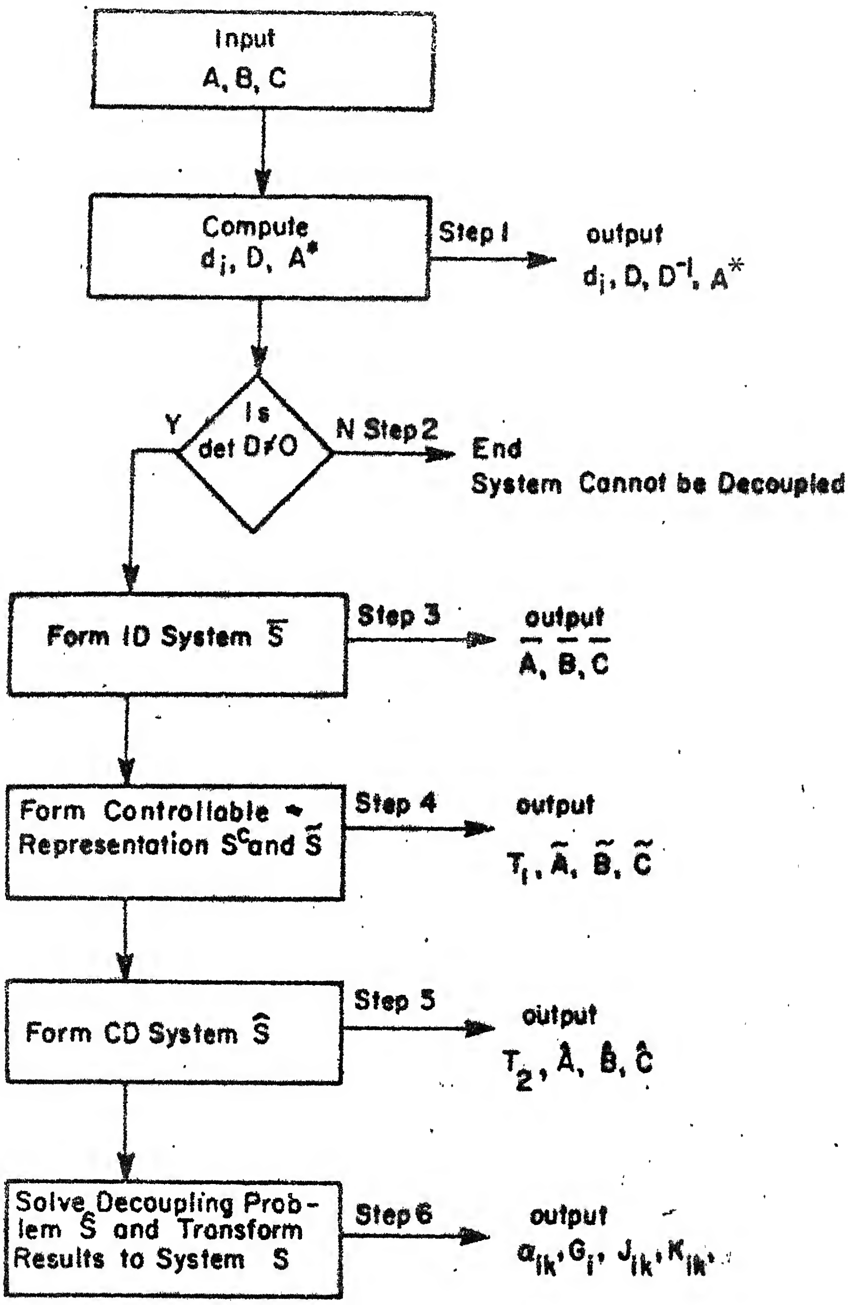


Fig. 3.2 Flow diagram for main steps in decoupling program.

$\tilde{S}$ is taken to be $\bar{S}$. If $n_c < n$, an $n \times n$ matrix $L = [L_1 \ L_2]$ is formed where L_1 is a set of linearly independent columns of $\bar{H}$ and L_2 are the columns forming basis for the null space of $\bar{H}'$, i.e., they are orthogonal to the columns of L_1. Then, with $T_1 = L^{-1}$, the transformation

$$\tau_2 : \tilde{A}\, T_1 = T_1 \bar{A}, \quad \tilde{B} = T_1 \bar{B}, \quad \tilde{C} T_1 = \bar{C}$$

would bring the system S in the form given by Lemma 3. Note that,

$$\tilde{F}\, T_1 = \bar{F}, \quad \tilde{G} = \bar{G} \qquad (3.19)$$

shows that $\tilde{S}$ and $\bar{S}$ are CLE.

Subroutine MFGR is used to find the rank, the independent columns and the orthogonal columns.

Step 5 : Transformation to CD Form ($\hat{S}$)

This is done by first calculating Q_i matrices from the S^c portions of $\tilde{S}$. Form C_i^c, $C_i^c A_c$, ..., $C_i^c (A_c)^{d_i}$. This forms part of Q_i. The $q_{\sigma_i} \ldots q_{\rho_i}$ are determined in the following way.

Form the controllability matrix H^c for S^c. From H^c form the matrices H_i, $i = 1, \ldots, m$ where H_i has the same columns as H^c except for n_c columns $i, i+m, \ldots, i+(n_c-1)m$ which are zero columns. Then $\eta H_i = 0$ is solved by subroutine MFGR producing $q_{\sigma_i} \ldots q_{\rho_i}$, which are chosen to be orthogonal to each other. That is, a basis for the (column) null space of the matrix

$$\begin{bmatrix} H_i{}' \\ C_i^c \\ \vdots \\ C_i^c (A_c)^{d_i} \end{bmatrix}$$

is obtained and the rows $q_i^1 \ldots q_i^{p_i - d_i - 1}$ for $i = 1, \ldots, m$ are obtained by taking the transpose of these basis columns.

Then $Q_{m+1} = \begin{bmatrix} q_{m+1}^1 \\ \vdots \\ q_{m+1}^{p_{m+1}} \end{bmatrix}$ is calculated, if necessary.

Since, the rows of Q_{m+1} must extend the rows of $Q_1, \ldots, Q_m$ to form a set of n_c linearly independent rows, the same are found by using subroutine MFGR with $[Q_1 \ \ldots \ Q_m]'$. The rows of Q_{m+1} are the transpose of the resulting basis columns.

Application of subroutine MFGR also determines $p_1, \ldots, p_m, p_{m+1} \ (= r_{m+1})$.

T_2 is, therefore, structured and the transformation,

$$\tau_3 : \hat{A} T_2 = T_2 \tilde{A}, \quad \hat{B} = T_2 \tilde{B}, \quad \hat{C} T_2 = \tilde{C}$$

yields the CD form $\hat{S}$, such that

$$\hat{F} T_2 = \tilde{F}, \qquad \hat{G} = \tilde{G} \tag{3.20}$$

<u>Step 6 :</u> Determination of the Synthesis Formulae

Since $\hat{S}$ is CIE to S, carrying $\{\hat{F}, \hat{G}\}$ back through (3.20), (3.19) and (3.17) gives $\{F, G\}$ for the decoupling of original system S.

Form of $\{\hat{F},\hat{G}\}$ and CD system (Theorem 5) suggests that $h_i(s,F,G)$ correspond to applying control law $\{\theta_i, \lambda_i\}$ to $S_i = \{A_i,b_i,c_i\}$. For $\theta_i = V_i(\sigma_i-\pi_i)$, the $\pi_i(\pi_{i1},\ldots \pi_{ip_i})$ and hence α_{ij}, the coefficients in the characteristic polynomial for each A_i, are determined by using Levarrier's algorithm. The K matrix (single variable theory) is formed and inverted to obtain V_i. When λ_i and θ_i are carried back through the CIE between $\hat{S}$ and S, eqns. (3.15) and related data are obtained. The matrices M_k^i in the print-out are given by expression of feedback matrix F of Lemma 4. The matrices G_i and J_k^i are calculated as indicated on the flow-charts shown in Appendix C.

3.3 AIRCRAFT LATERAL DYNAMICS: STATEMENT OF PROBLEM

The linear dynamical equation for aircraft lateral motion was derived in Chapter II as,

$$\dot{\underline{x}} = A\,\underline{x} + B\,\underline{u}$$

$$\underline{y} = C\,\underline{x}$$

where $\underline{u}$ and $\underline{y}$ were defined as two-component input and output vectors, respectively.

The transfer function matrix of this system is

$$H(s) = C(sI - A)^{-1}B$$

where,

$$
sI-A = \begin{bmatrix} s-\frac{\bar{q}S}{mU_1}\,C_{y_\beta} & \frac{g}{U_1} & 0 & -1 \\ 0 & s & 1 & 0 \\ \frac{\bar{q}Sb}{I_{xx}}\,C_{l_\beta} & 0 & s & 0 \\ \frac{\bar{q}Sb}{I_{zz}}\,C_{n_\beta} & 0 & 0 & s \end{bmatrix}
$$

The determinant of the above matrix is

$$
\Delta(s) = s^4 + \frac{\bar{q}Sb}{I_{zz}}\,C_{n_\beta}\cdot s^2 + \left(\frac{g}{U_1}\cdot\frac{\bar{q}Sb}{I_{xx}}\,C_{l_\beta} - \frac{\bar{q}S}{mU_1}\,C_{y_\beta}\right)s
$$

$$
= s^4 + Ps^2 + Qs \qquad \text{(say)}
$$

$$
= s(s+\alpha_2)(s+\alpha_3)(s+\alpha_4) \qquad \text{(say)}
$$

It can be seen that

$$
C(sI-A)^{-1}B = \frac{1}{\Delta(s)}\begin{bmatrix} \frac{g}{U_1}\frac{\bar{q}Sb}{I_{xx}}\,C_{l_{\delta_A}}\cdot s + \frac{\bar{q}Sb}{I_{zz}}\,C_{n_{\delta_A}}\cdot s^2 & \frac{g}{U_1}\frac{\bar{q}Sb}{I_{xx}}\,C_{l_{\delta_R}}\cdot s + \frac{\bar{q}Sb}{I_{zz}}\,C_{n_{\delta_R}}\cdot s^2 \\ -\left(s^2 - \frac{\bar{q}S}{mU_1}\,C_{y_\beta}\cdot s + \frac{\bar{q}Sb}{I_{zz}}\,C_{n_\beta}\right)\frac{\bar{q}Sb}{I_{xx}}\,C_{l_{\delta_A}} + \frac{(\bar{q}Sb)^2}{I_{xx}I_{zz}}\,C_{n_{\delta_A}}\cdot C_{l_\beta} & -\left(s^2 - \frac{\bar{q}S}{mU_1}\,C_{y_\beta}\cdot s + \frac{\bar{q}Sb}{I_{zz}}\,C_{n_\beta}\right)\frac{\bar{q}Sb}{I_{xx}}\,C_{l_{\delta_R}} + \frac{(\bar{q}Sb)^2}{I_{xx}I_{zz}}\,C_{l_\beta}\cdot C_{n_{\delta_R}} \end{bmatrix}
$$

$$= \frac{1}{\Delta(s)} \begin{bmatrix} Rs^2+Ts & Vs^2+Ws \\ R's^2+T's+K & V's^2+W's+K' \end{bmatrix} \quad \text{(say)}$$

Therefore,

$$H(s) = \begin{bmatrix} \dfrac{Rs+T}{s^3+Ps+Q} & \dfrac{Vs+W}{s^3+Ps+Q} \\[2ex] \dfrac{R's^2+T's+K}{s^4+Ps^2+Qs} & \dfrac{V's^2+W's+K'}{s^4+Ps^2+Qs} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

where the values of $P,Q,R,T,V,W,R',T',K,V',W'$, and K' can be determined from any aircraft configuration pertaining to a flying condition.

So, inputs and outputs are coupled as indicated by the transfer function matrix, i.e.,

$$\varphi = h_{11}\ \delta_A + h_{12}\ \delta_R$$

$$\beta = h_{21}\ \delta_A + h_{22}\ \delta_R$$

So, the statement of the decoupling problem pertaining to aircraft lateral dynamics is : <u>find a suitable control law $\{F,G\}$ which decouples φ and β.</u> The same practical questions arise with respect to this problem :

i) is it possible to find a control law which decouples,

ii) if decoupling is possible what freedom of choice for $\{F,G\}$ exists,

iii) what kind of closed-loop characteristics are attainable,

iv) once an attainable closed-loop characteristics is specified, how is $\{F,G\}$ determined.

All these questions will be answered once the data derivable from matrices A,B and C pertaining to a particular aircraft and flying condition is determined by running the above explained computer program.

This would be taken up in the next chapter.

CHAPTER IV

APPLICATION OF DECOUPLING THEORY
(Achievement of Stable Decoupled Dynamics)

This chapter aims at illustrating the application of the decoupling theory explained previously in arriving at numerical solution of the lateral dynamics decoupling problem as applied to a high-performance aircraft similar to X-15 that was produced by North American Company as a research airplane for NASA. The aircraft configuration and aerodynamic characteristics under stated flying conditions as taken from a NASA Report [6] and relevant to our problem are given in Table 4.1. By incorporating the values of parameters from the computer output into the relations stated in the last chapter, complete synthesis of the decoupling problem, raised therein, is obtained. Step inputs of aileron and rudder deflections are simulated and the respective φ and β time responses in both open-loop and closed-loop cases have been drawn.

4.1 COMPUTER INPUTS

The A and B matrices, computed from (2.47), (2.48) and the data of Table 4.1, are

$$A = \begin{bmatrix} -0.0176076 & 0.005367 & 0.3490658^{**} & -1 \\ 0 & 0 & 1 & 0 \\ 2.6719776 & 0 & 0 & 0 \\ 4.1260132 & 0 & 0 & 0 \end{bmatrix}$$

** See foot-note next page.

Table 4.1 Aerodynamic Characteristics of an Advanced Airplane Similar to X-15

b,ft	.....	.....	22.36
g,ft/sec^2	.....	.....	32.2
I_{xx},slug -ft^2	.....	.....	5,021
I_{zz},slug -ft^2	.....	.....	67,199
m,slugs	.....	.....	390.4
$\bar{q}$,lb/sq ft	.....	.....	200
S,sq ft	.....	.....	200
U_1,ft/sec	.....	.....	6,000
Altitude,ft	.....	.....	125,000
Mach number	.....	.....	6
α_1,deg	.....	.....	20
C_{l_β},per radian	.....	.....	0.015
C_{n_β},per radian	.....	.....	0.31
C_{y_β},per radian	.....	.....	-1.0
$C_{l_{\delta_A}}$,per radian	.....	.....	-0.075
$C_{l_{\delta_R}}$, per radian	.....	.....	-0.15
$C_{n_{\delta_A}}$,per radian	.....	.....	0.08
$C_{n_{\delta_R}}$,per radian	.....	.....	-0.108

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -13.359888 & -26.719776 \\ -1.0647776 & -1.4374497 \end{bmatrix}$$

The matrix C was already fixed in (2.49) as

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Integers N (= 4) and M (= 2) are fed in as the sizes of the state and the input (and output) vectors respectively.

4.2 SYNTHESIS

The computer print-out (attached as Appendix D) obtained after running the program for the design of decoupled system explained previously is referred. In connection with determination of a control law $\{F,G\}$ which decouples, the four questions raised in Sec. 3.3 are now answered.

i) Since matrix D, uniquely determined by the dynamics of the lateral motion, i.e., A,B,C , is found to be non-singular, necessary condition of decoupling is satisfied. Hence, it is possible to find a control law that would decouple φ and β responses.

** This element of matrix A while, deriving the equations of motion had yielded zero by neglecting the nonlinear term wp in eqn. (ii) of Table 2.2. Its inclusion gives rise to the term α in A whose value under the stated flying conditions is appreciable and hence may not be neglected. Though α, primarily concerns longitudinal motion, some inertial coupling is normally found in case of high-performance aircrafts.

ii) The computer output gives

$$\begin{aligned} &\text{D-SUB-1, i.e.,} && d_1 = 1 \\ &\text{D-SUB-2, i.e.,} && d_2 = 1 \\ &\text{P-SUB-1, i.e.,} && p_1 = 2 \\ &\text{P-SUB-2, i.e.,} && p_2 = 2 \end{aligned}$$

The fact that P-SUB-3 does not appear in the computer output signifies that the completely unobservable subsystem found in the structure of canonically decoupled form is not present. This, in fact, would happen in most of the cases. Thus, $p_3 = 0$.

Therefore, r_i indices would get,

$$\begin{aligned} r_1 &= 0 && \text{using } r_i = p_i-1-d_i \\ r_2 &= 0 && \\ r_3 &= 0 && (\text{By def., } \because\ r_3 = p_3) \\ \text{and} \quad r_4 &= 0 && (\text{By def., } \because \text{ the open-loop sys. is controllable}). \end{aligned}$$

Because, $r_i = 0$ for $i = 1, \ldots, (m+2)$, i.e., 4, by Theorem 5 part (i) of Chapter III, the polynomials, $\alpha_i(s) = 1$ for $i = 1, \ldots, 4$.

The class of $\{F, G\}$ which decouples as given by relations (3.15) would be

$$G = \lambda_1 G_1 + \lambda_2 G_2 = \begin{bmatrix} 0.16556\ \lambda_1 & -0.56069\ \lambda_2 \\ -0.12020\ \lambda_1 & 0.28035\ \lambda_2 \end{bmatrix}$$

and

$$F = -D^{-1}A^{*}+(\sigma_{11}-\pi_{11})J_{11}+(\sigma_{12}-\pi_{12})J_{12}+(\sigma_{21}-\pi_{21})J_{21}+(\sigma_{22}-\pi_{22})J_{22}$$

(F^u is not defined as system is controllable).

$$= \begin{bmatrix} -2.23266 - 0.00683\,\sigma_{21} + 5.59530\,\sigma_{22} & -0.00005 - 0.00397\,\sigma_{12} + 0.00208\,\sigma_{21} & -0.00044 - 0.12046\,\sigma_{11} + 0.13544\,\sigma_{21} & 0.00987 - 0.38800\,\sigma_{21} \\ 1.21633 + 0.00342\,\sigma_{21} - 2.79770\,\sigma_{22} & 0.00003 + 0.00288\,\sigma_{12} - 0.00104\,\sigma_{21} & 0.00022 + 0.08746\,\sigma_{11} - 0.06772\,\sigma_{21} & -0.00494 + 0.19400\,\sigma_{21} \end{bmatrix}$$

λ_1, λ_2, σ_{11}, σ_{12}, σ_{21}, σ_{22} being arbitrary real numbers denote the freedom of choice with F and G. This freedom enables the designer to place the closed-loop poles, theoretically, anywhere in the complex s-plane.

iii) Elements of the diagonal matrix H(s,F,G) from (3.14a) of Theorem 5 are,

$$h_i(s,F,G) = \frac{\lambda_i}{\Psi_i(s,\sigma_i)} \qquad i = 1,2$$

where the polynomials Ψ_1 and Ψ_2 would be

$$\Psi_1 = s^2-\sigma_{11}s-\sigma_{12}, \qquad \Psi_2 = s^2-\sigma_{21}s-\sigma_{22}$$

Therefore,

$$H(s,F,G) = \begin{bmatrix} \dfrac{\lambda_1}{s^2-\sigma_{11}s-\sigma_{12}} & 0 \\ 0 & \dfrac{\lambda_2}{s^2-\sigma_{21}s-\sigma_{22}} \end{bmatrix} \tag{4.1}$$

With λ's and σ's to be chosen arbitrarily, the above tells the designer what closed-loop characteristics are attainable. Once these are selected, a particular set of $h_i(s,F,G)$ is specified and the corresponding control law $\{F,G\}$ determined. This follows,

The open-loop poles** of the unaugmented aircraft in question whose lateral dynamics are coupled and unstable are found to be

$$\alpha_1 = 0$$
$$\alpha_2 = 0.00449$$
$$\alpha_3 \text{ and } \alpha_4 = -0.01105 \pm j\, 1.78698$$

The desired stability is represented by the roots of the controlled system which were chosen to satisfy the military handling qualities specifications [18]. These were selected as follows :

$$\alpha_1 = -0.0346$$
$$\alpha_2 = -0.693$$
$$\alpha_3 \text{ and } \alpha_4 = -0.346 \pm j\, 3.14$$

**Given by the eigenvalues of A, these are calculated by using IMSL subroutine EIGRF.

Comparison with roots of the basic (unaugmented) airplane shows : (a) that the lateral motions of the airplane are now completely stable; and (b) that the damping ratio of the complex pole pair is much higher for the controlled system.

Since, the roll motion of the aircraft is non-oscillatory by nature whereas the yaw motion is oscillatory, the simple poles selected above will be attributed to φ response while the complex poles will be associated with the β response. The selection of roots and their appropriate assignment to the φ or β response also bears testimony to the fact that while, on one hand, the resulting ratio $\frac{\varphi}{\delta_A} = \frac{1}{(s+0.0346)(s+0.693)}$ lies fairly close to $\frac{1}{(s+a)(s+b)}$, with $a \approx 0^-$ and $b >> a$, which is recommended by design considerations, a time period of 2 secs. given by the complex roots above, on the other hand, is considered fairly acceptable for the transients of β-response.

With desired stability, our system would, therefore, be described by

$$H(s,F,G) = \begin{bmatrix} \frac{\lambda_1}{(s+0.0346)(s+0.693)} & 0 \\ 0 & \frac{\lambda_2}{(s+0.346-j3.14)(s+0.346+j3.14)} \end{bmatrix} \tag{4.2}$$

which, by comparing with (4.1), yields

$$\sigma_{11} = -0.7276 \qquad \sigma_{12} = -0.0239778$$
$$\sigma_{21} = -0.692 \qquad \sigma_{22} = -9.979316 \tag{4.3}$$

iv) If $\lambda_1 = \lambda_2 = 1$, arbitrarily (4.4)

then, in terms of value parameters in (4.3) and (4.4), the control law $\{F, G\}$ which results in a stable decoupled lateral response gets fixed up and takes the form,

$$F = \begin{bmatrix} 3.3548126 & -0.0019437 & 0.0145368 & -0.3781974 \\ -1.5779528 & 0.0018741 & 0.0199584 & 0.1890622 \end{bmatrix}$$

and

$$G = \begin{bmatrix} 0.16556 & -0.56069 \\ -0.12020 & 0.28035 \end{bmatrix}$$

A similar set of gains can be computed for any set of desired roots. In a complete analysis, the roots may have to be chosen selectively to insure reasonable gains.

The system matrix of the closed-loop system is found to be,

$$A+BF = \begin{bmatrix} -0.01761 & 0.00537 & 0.34907 & -1.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.014601 & -0.0241079 & -0.7273205 & 0.0009735 \\ 9.9663755 & -0.0047635 & -0.0132108 & -0.6748597 \end{bmatrix}$$

Its eigenvalues and hence the roots of the characteristic equation are found to be

$$-0.0346226, \quad -0.6921571, \quad -0.34620 \pm j3.138997$$

which tally with the chosen roots. This corroborates our synthesis of decoupling and stabilising controllers.

4.3 SIMULATION

Having obtained mathematical models of the system, in both open-loop and closed-loop cases, we are now in a position to obtain their time responses by simulating any specific input signal (say a unit-step).

Referring to Sec. 3.3 and Table 4.1, the open-loop transfer function matrix of the original system, after computation of various coefficients involved therein, would be given by

$$H(s) = \frac{1}{\Delta(s)} \begin{bmatrix} 1.06478s^2 - 0.0717s & -1.43745s^2 - 0.1434s \\ -13.35988s^2 + 0.23524s + 57.96814 & 26.71978s^2 + 110.24614s - 3.84081 \end{bmatrix}$$

where $\Delta(s) = s(s-0.00449)(s+0.01105-j1.78698)(s+0.01105+j1.78698)$

If a unit step input of aieleron deflection (i.e., $\delta_A = \frac{1}{s}$) is now simulated on the system, the resulting φ and β responses would be given by

$$\varphi = \frac{1.06478s - 0.0717}{s(s-0.00449)(s+0.01105-j1.78698)(s+0.01105+j1.78698)}$$

and

$$\beta = \frac{-13.35988s^2 + 0.23524s + 57.96814}{s^2(s-0.00449)(s+0.01105-j1.78698)(s+0.01105+j1.78698)}$$

If, on the other hand, the system is excited by a unit-step of rudder reflection (i.e. $\delta_R = \frac{1}{s}$), the corresponding φ and β responses would become,

$$\varphi = \frac{-1.43745s - 0.1434}{s(s-0.00449)(s+0.01105-j1.78698)(s+0.01105+j1.78698)}$$

and

$$\beta = \frac{26.71978s^2 + 110.24614s - 3.84081}{s^2(s-0.00449)(s+0.01105-j1.78698)(s+0.01105+j1.78698)}$$

From (4.2), the transfer function matrix of the finally obtained closed-loop system is given by

$$H(s,F,G) = \begin{bmatrix} \frac{1}{(s+0.0346)(s+0.693)} & 0 \\ 0 & \frac{1}{(s+0.346-j3.14)(s+0.346+j3.14)} \end{bmatrix}$$

For the unit-step aileron deflection, the output response of the system is

$$\varphi = \frac{1}{s(s+0.0346)(s+0.693)}$$

And, the output response for the unit-step rudder deflection would be,

$$\beta = \frac{1}{s(s+0.346-j3.14)(s+0.346+j3.14)}$$

A complete time response expression for each of the above responses in frequency-domain can then be obtained by splitting

them into partial fractions and then carrying out Laplace transform inversion under zero initial conditions. This was achieved by writing a computer program.

Time-responses drawn have been shown in the adjoining figures. The open-loop responses (Fig. 4.1 to Fig. 4.4) are oscillatory by nature and indicate that the stability of lateral dynamics in the case of unaugmented aircraft is unsatisfactory. The decoupled φ and β responses are shown in Figs. 4.5 and 4.6 respectively and have been made completely stable. Whereas the closed-loop β response is found to have a settling time (t_s) of 4.5 secs. approx., the φ response turns out to be slightly sluggish, which, if desired, can be improved by manipulating the dominant root.

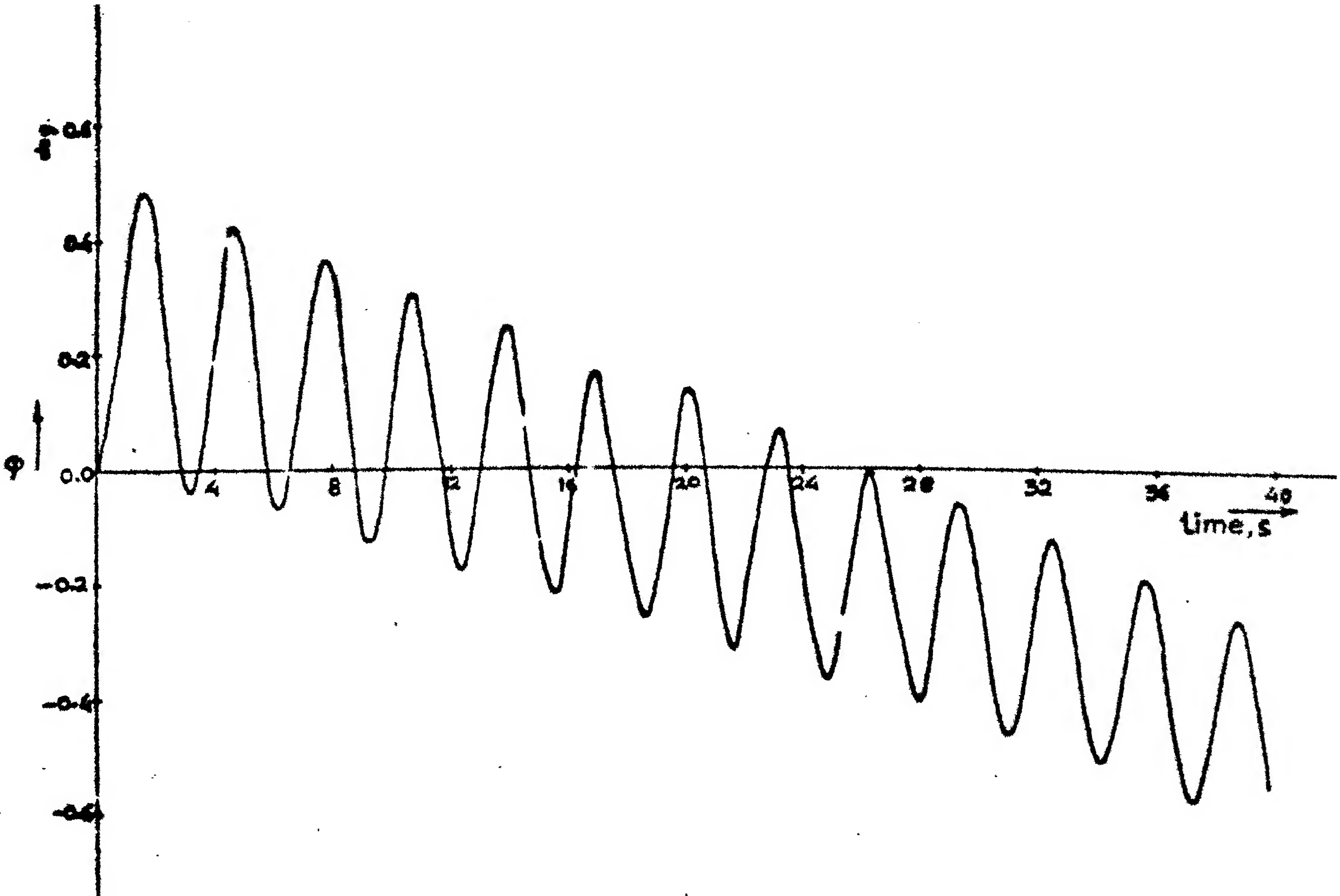


Fig. 4.1 Step response for φ (input δ_A)

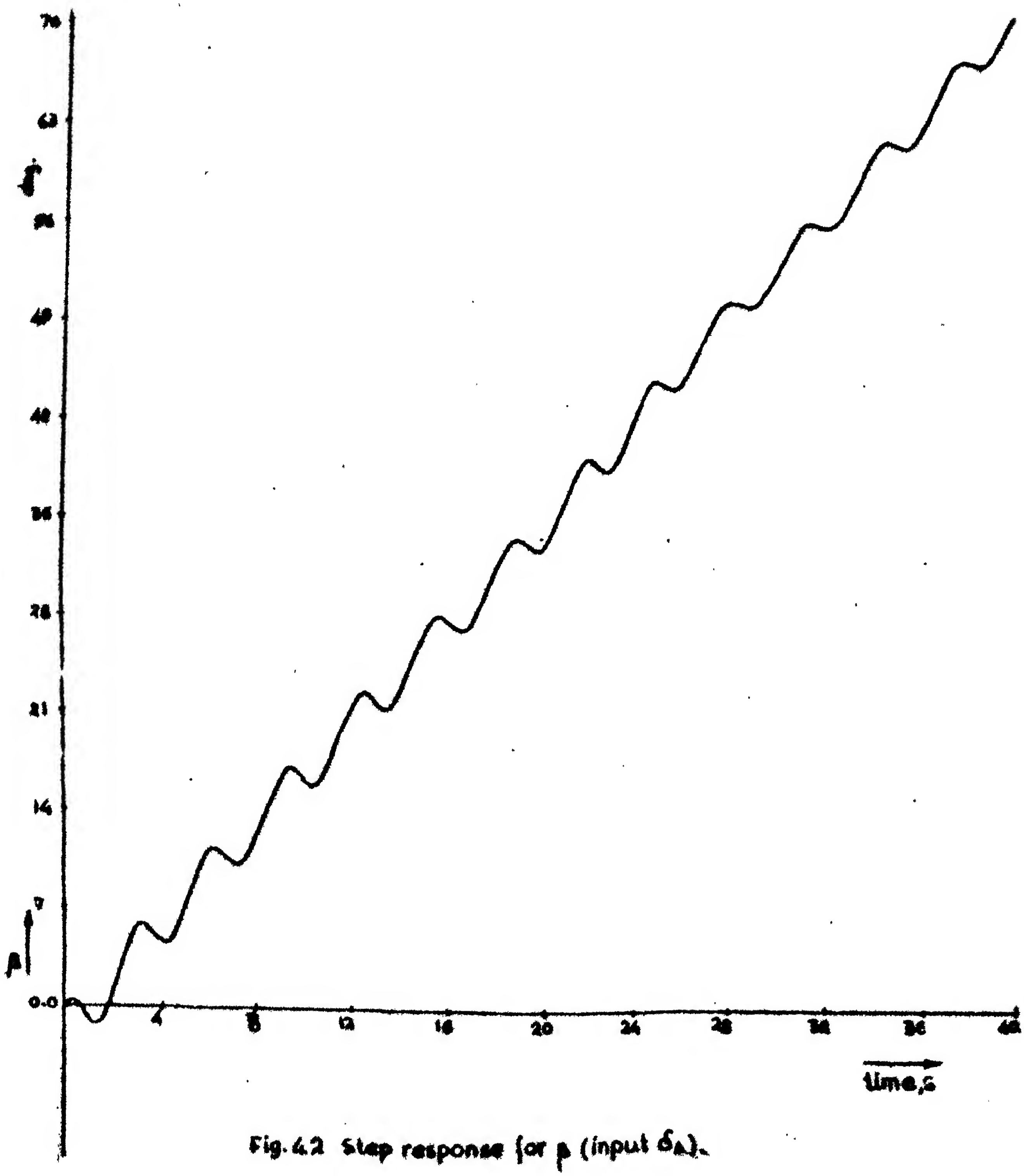


Fig. 4.2 Step response for β (input δ_A).

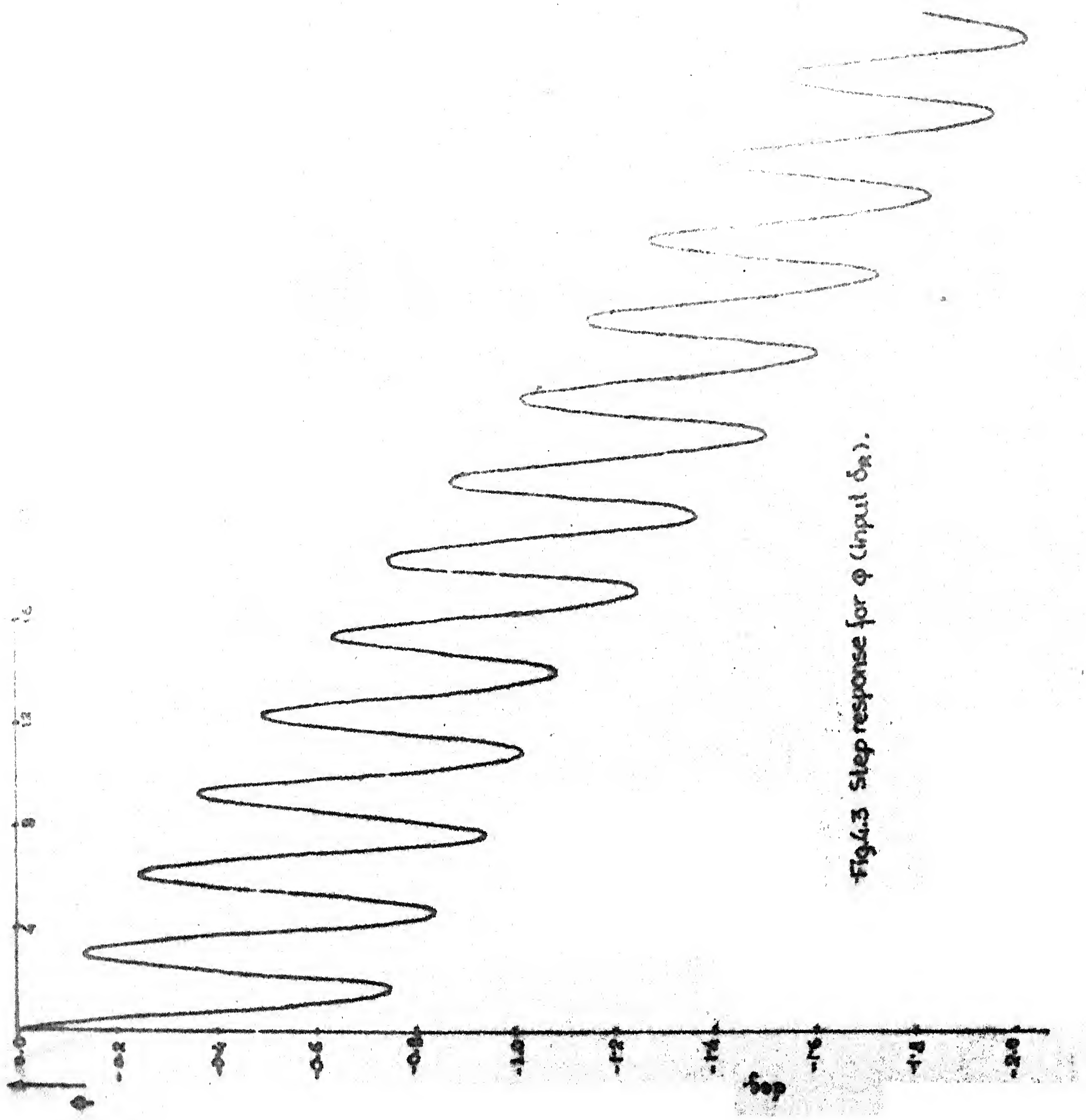


Fig.4.3 Step response for φ (input δ_R).

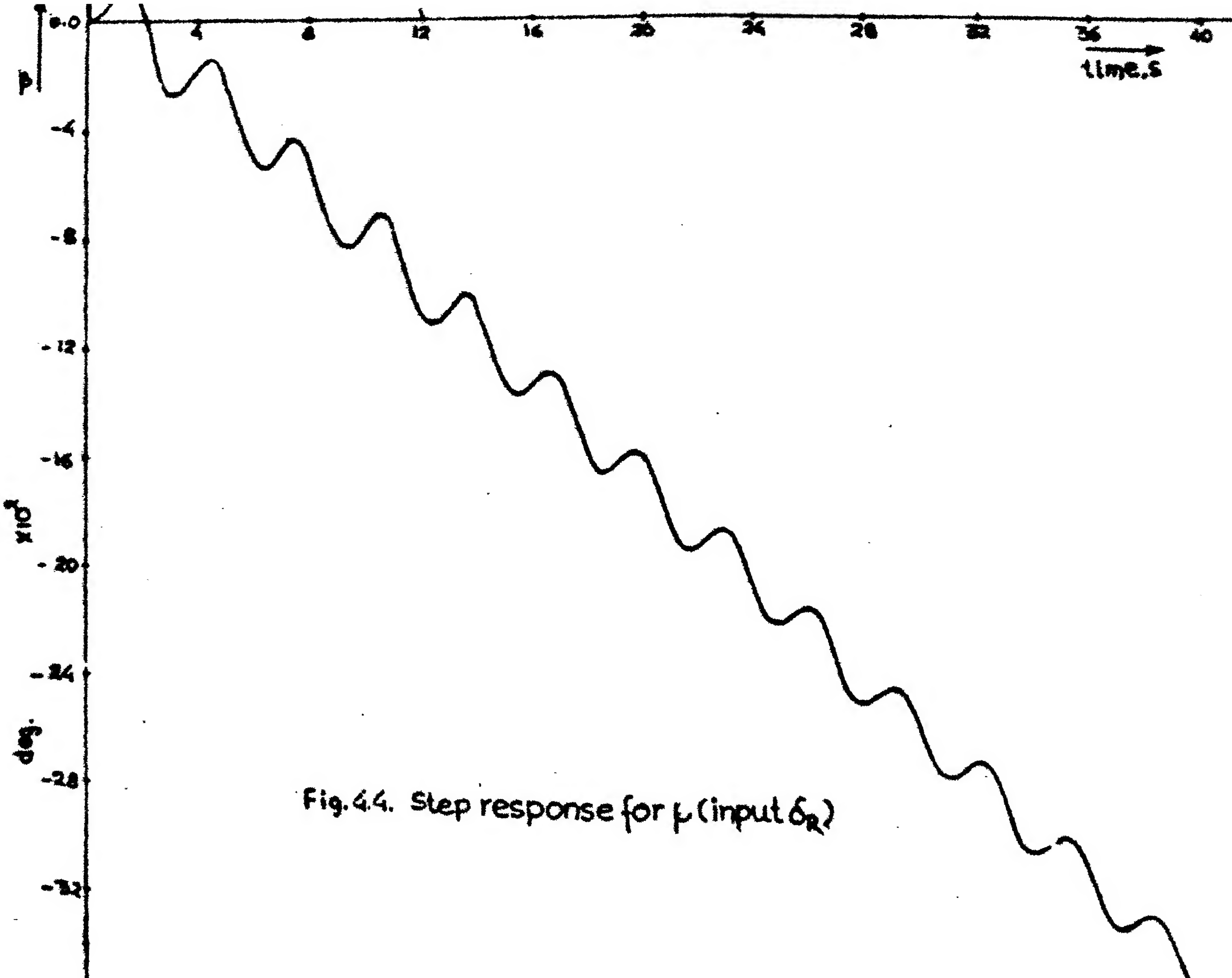


Fig. 4.4. Step response for μ (input δ_R)

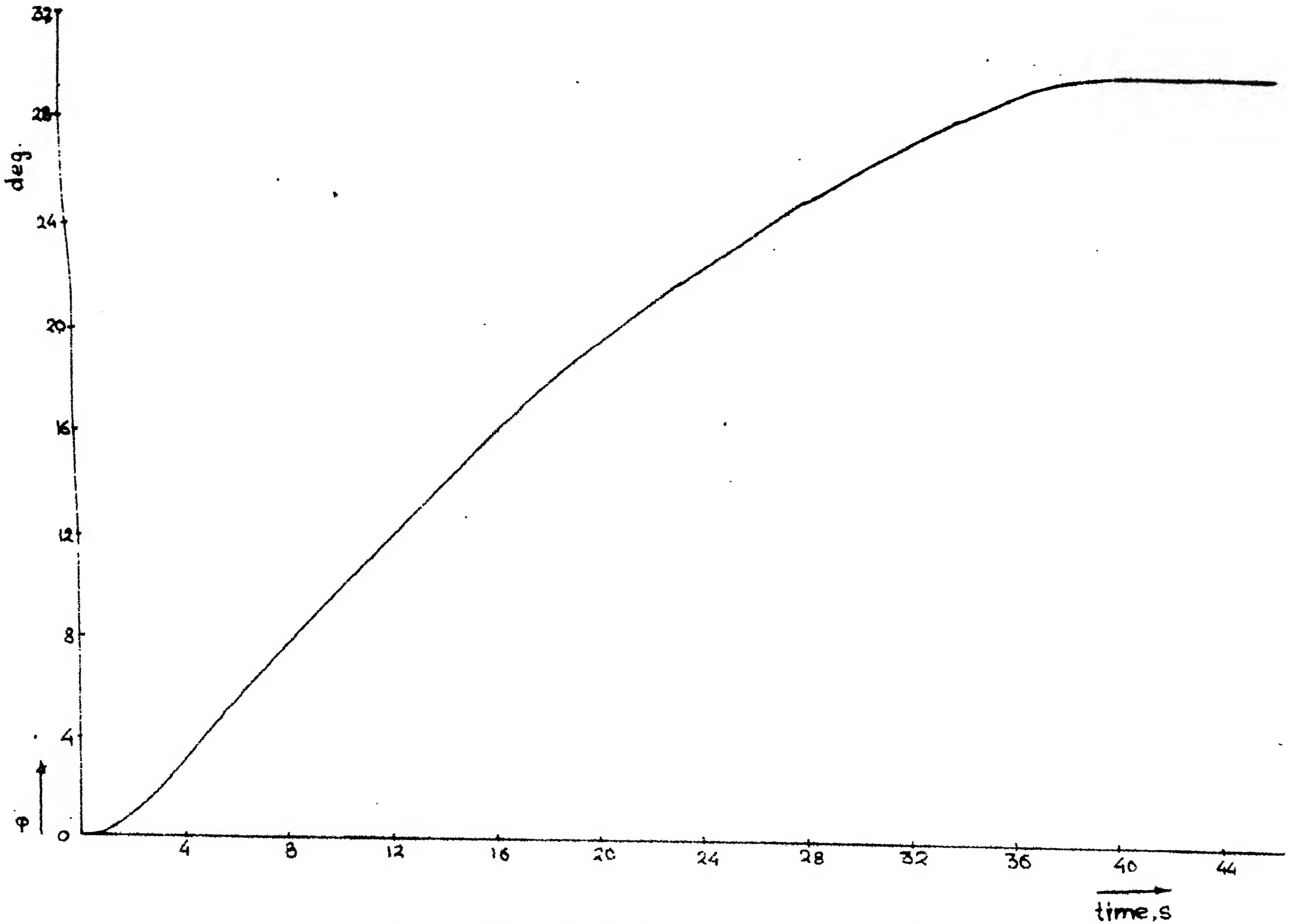


Fig. 4.5. Closed loop step response for φ (input δ_A)

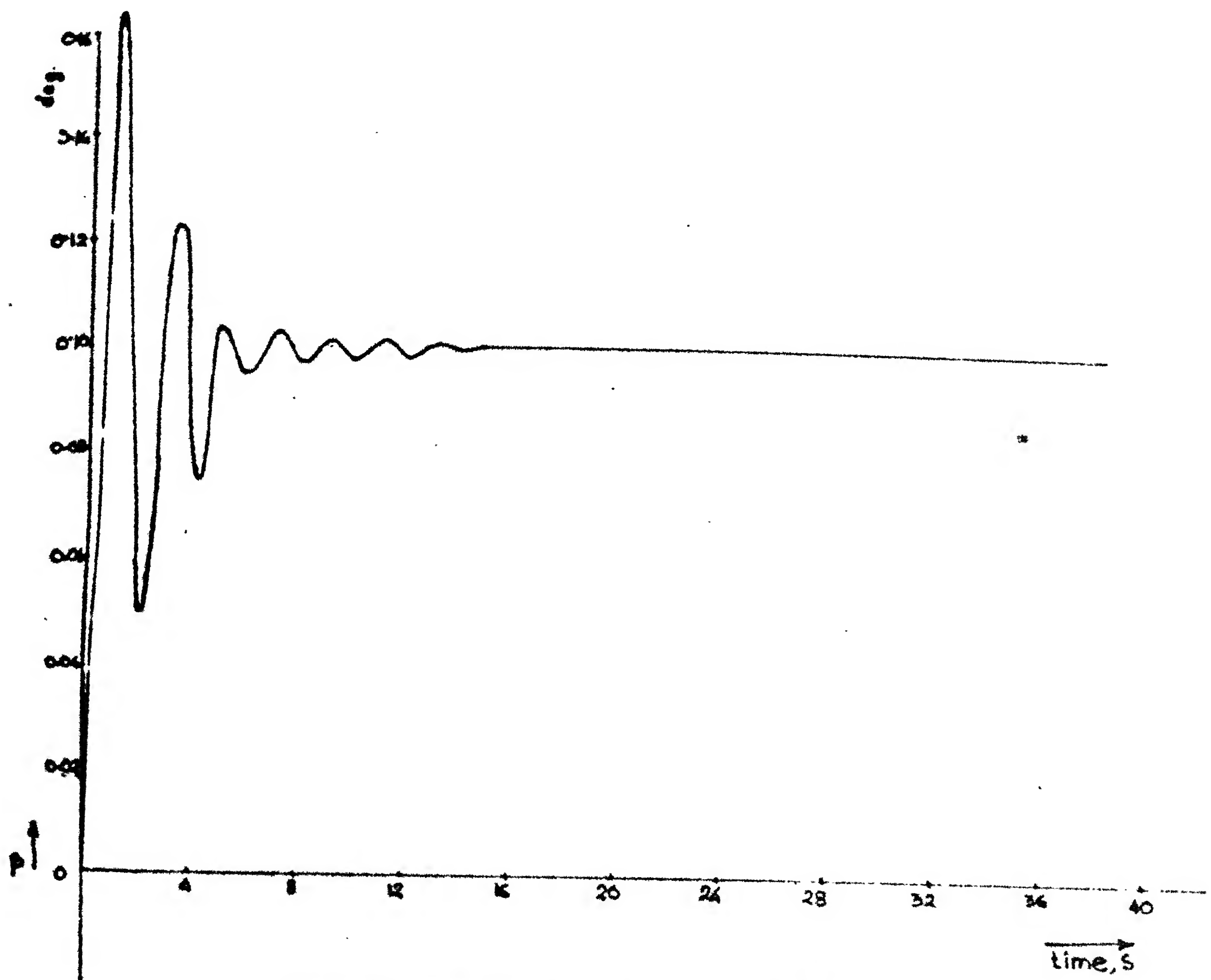


Fig. 4.6 Closed loop step response for β (input δ_R)

CHAPTER V

CONCLUSIONS

For the aircraft lateral control problem considered, Gilbert's comprehensive theory for the decoupling of multivariable systems by state feedback provides a constructive procedure to determine the classes of feedback and interconnect control matrices that give a decoupled structure. His algorithm, translated into machine language, is readily adaptable to any problem. It is felt that this scheme would provide a better synthesis procedure than other methods because of the relative ease of solution of the pole placement problem.

In arriving at the solution of the problem, our treatment does not take into account the transfer function due to electro-hydraulic power devices (comprising of servo-valve, hydraulic jack, motor etc) demanding deflection of control surfaces. As, in practice, this forms part of the control loop, its inclusion, that changes dynamics of the system and hence its system matrix, would pose no problem when obtaining the solution because of the generality of the method.

While deriving the final form of equations of motion, control derivatives C_{l_p} and C_{n_r} were neglected for simplicity which, in fact, is slightly unusual. Much as we would have

liked to retain these terms, their numerical values did not exist in the aircraft data. But their inclusion would only replace one of the zero elements in the system matrix by a finite value and, therefore, would, in no way, prevent us in achieving synthesis.

Whereas in our case, the free elements (σ's) of the feedback matrix (F) were fixed by specifying closed loop poles pertaining to desirable military handling qualities, the problem can be tackled alternately [4b] in a way that would optimise these free parameters with respect to a specific criterion like minimising a cost functional involving error and control amplitudes. This amounts to chosing σ_i by solving an optimisation problem for S_i.

The effect of parameter variations and disturbance inputs on the aircraft system have not been considered as the method used makes no provision to take them into account. In practice, the system, which is decoupled for nominal parameter values, may not remain decoupled when parameters undergo variations. However, if the variation are not large, the system may be assumed to behave in an'approximately decoupled' fashion.

inv. desirable -

APPENDIX A

CONTROL AND STABILITY DERIVATIVES

The control and stability derivatives involved in eqn. (2.38) are defined as follows. Their usual dimensions are deg^{-1} or rad^{-1}.

Symbol	Definition
$C_{l_\beta} = \frac{\partial C_l}{\partial \beta}$	Variation of rolling moment coefficient with sideslip angle (i.e. dihedral angle).
$C_{l_{\delta_A}} = \frac{\partial C_l}{\partial \delta_A}$	Variation of rolling moment coefficient with aileron angle (i.e. lateral control power).
$C_{l_{\delta_R}} = \frac{\partial C_l}{\partial \delta_R}$	Variation of rolling moment coefficient with rudder angle.
$C_{y_\beta} = \frac{\partial C_y}{\partial \beta}$	Variation of side force coefficient with sideslip angle.
$C_{y_{\delta_R}} = \frac{\partial C_y}{\partial \delta_R}$	Variation of side force coefficient with rudder angle.
$C_{y_{\delta_A}} = \frac{\partial C_y}{\partial \delta_A}$	Variation of side force coefficient with aileron angle.
$C_{n_\beta} = \frac{\partial C_n}{\partial \beta}$	Variation of yawing moment coefficient with sideslip angle.
$C_{n_{\delta_R}} = \frac{\partial C_n}{\partial \delta_R}$	Variation of yawing moment coefficient with rudder angle. (Rudder power)
$C_{n_{\delta_A}} = \frac{\partial C_n}{\partial \delta_A}$	Variation of yawing moment coefficient with aileron angle.

Symbol	Definition
$C_{y_p} = \dfrac{\partial C_y}{\partial(\frac{pb}{2U_1})}$	Variation of side-force coefficient with roll rate.
$C_{l_p} = \dfrac{\partial C_l}{\partial(\frac{pb}{2U_1})}$	Variation of rolling moment coefficient with roll rate.
$C_{n_p} = \dfrac{\partial C_n}{\partial(\frac{pb}{2U_1})}$	Variation of yawing moment coefficient with roll rate.
$C_{y_r} = \dfrac{\partial C_y}{\partial(\frac{rb}{2U_1})}$	Variation of side-force coefficient with yaw rate.
$C_{l_r} = \dfrac{\partial C_l}{\partial(\frac{rb}{2U_1})}$	Variation of rolling moment coefficient with yaw rate.
$C_{n_r} = \dfrac{\partial C_n}{\partial(\frac{rb}{2U_1})}$	Variation of yawing moment coefficient with yaw rate.
$C_{N_{T_\beta}} = \dfrac{\partial C_{N_T}}{\partial \beta}$	Variation of thrust induced yawing moment coefficient (due to normal force) with sideslip angle.
$C_{y_{\dot\beta}} = \dfrac{\partial C_y}{\partial(\frac{\dot\beta b}{2U_1})}$	Variation of side-force coefficient with rate of change of sideslip angle.
$C_{l_{\dot\beta}} = \dfrac{\partial C_l}{\partial(\frac{\dot\beta b}{2U_1})}$	Variation of rolling moment coefficient with rate of change of sideslip angle.
$C_{n_{\dot\beta}} = \dfrac{\partial C_n}{\partial(\frac{\dot\beta b}{2U_1})}$	Variation of yawing moment coefficient with rate of change of sideslip angle.

APPENDIX B

CONTROL DERIVATIVES $C_{n_{\delta_A}}$ AND $C_{n_{\delta_R}}$

a) Yawing Moment due to Aileron ($C_{n_{\delta_A}}$) :

Consider a positive aileron deflection (Fig. B.1); on the starboard wing, where the aileron is down, the lift of the wing is increased, and so are the profile and induced drag. On the aileron-up wing, however, the lift and hence the induced drag component is reduced. Whereas + ΔL and - ΔL would cause a banking moment (as desired) that would starboard the aircraft, the difference in the drag acting on the two wings produces a yawing moment that would turn the aircraft to port side. For this reason it is described as an adverse aileron yaw.

If, on the other hand, the ailerons are being used to turn the aircraft to port, then the resulting yawing moment acts to starboard it as shown in Fig. B.2.

Values of $C_{n_{\delta_A}}$ are difficult to estimate and it is generally best to use experimental data. Its magnitude can vary widely from one airplane configuration to another.

b) Rolling Moment due to Rudder ($C_{l_{\delta_R}}$) :

A physical explanation of the aerodynamic mechanism by which rolling moment due to directional control is obtained is presented in Fig. B.3.

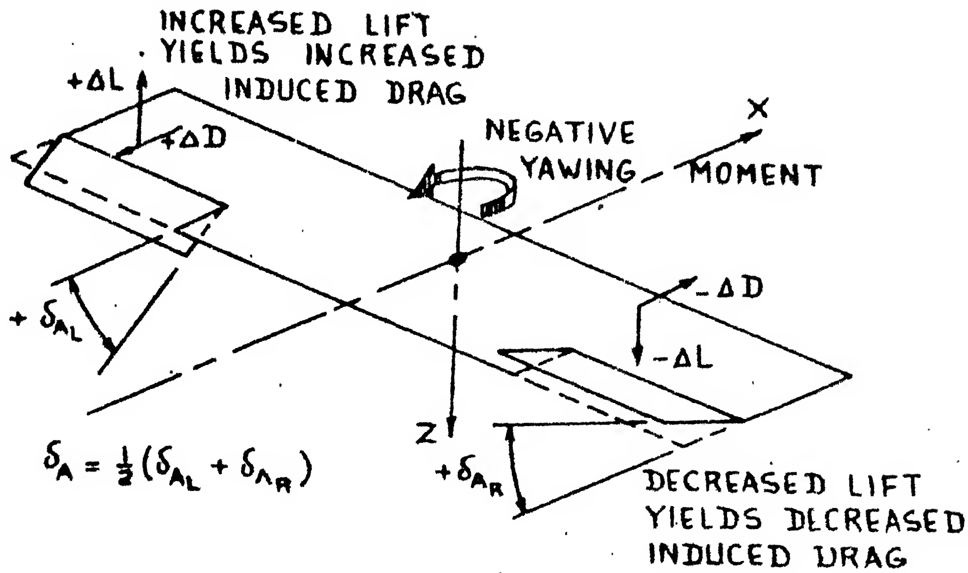


Fig. B.1 Physical Explanation of Yawing Moment due to Ailerons

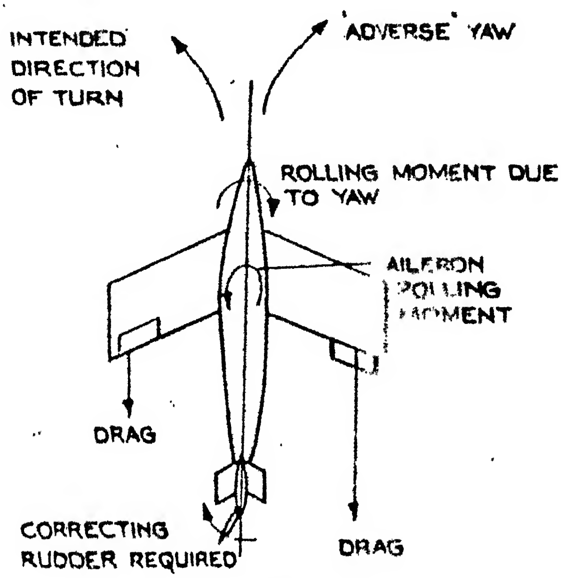


Fig. B. 2. Adverse aileron yaw

When the rudder is deflected, the resulting aerodynamic force 'F_{A_y}' on the vertical fin and rudder produces a required yawing moment about the C.G. of the aircraft through the distance arm X_{V_s} along the X_s - axis. Usually, this resultant force acts at a distance Z_{V_s} (along the Z_s-axis) above the longitudinal axis through the C.G. and consequently produces a rolling moment as well. Because, the purpose of the rudder is directional control, this rolling moment due to rudder must be seen as an undesirable side effect.

To develop an expression for $C_{l_{\delta_R}}$, associate the force due to rudder on the vertical tail with a 'side-force derivative', $C_{y_{\delta_R}}$, so that

$$F_{A_{y(\text{rudder})}} = C_{y_{\delta_R}} \cdot \delta_R \cdot \bar{q} S$$

Therefore, the rolling moment about the X_s (stability) axis is:

$$L_{(\text{rudder})} = F_{A_y} \cdot Z_{V_s}$$

$$= C_{y_{\delta_R}} \cdot \delta_R \cdot \bar{q} S \cdot Z_{V_s} = C_{l_{\delta_R}} \cdot \delta_R \cdot \bar{q} S b$$

(non-dimensionalised)

so that;

$$C_{l_{\delta_R}} = C_{y_{\delta_R}} \cdot \frac{Z_{V_s}}{b}$$

$C_{y_{\delta_R}}$ is normally known is terms of vertical tail area and lift curve slope.

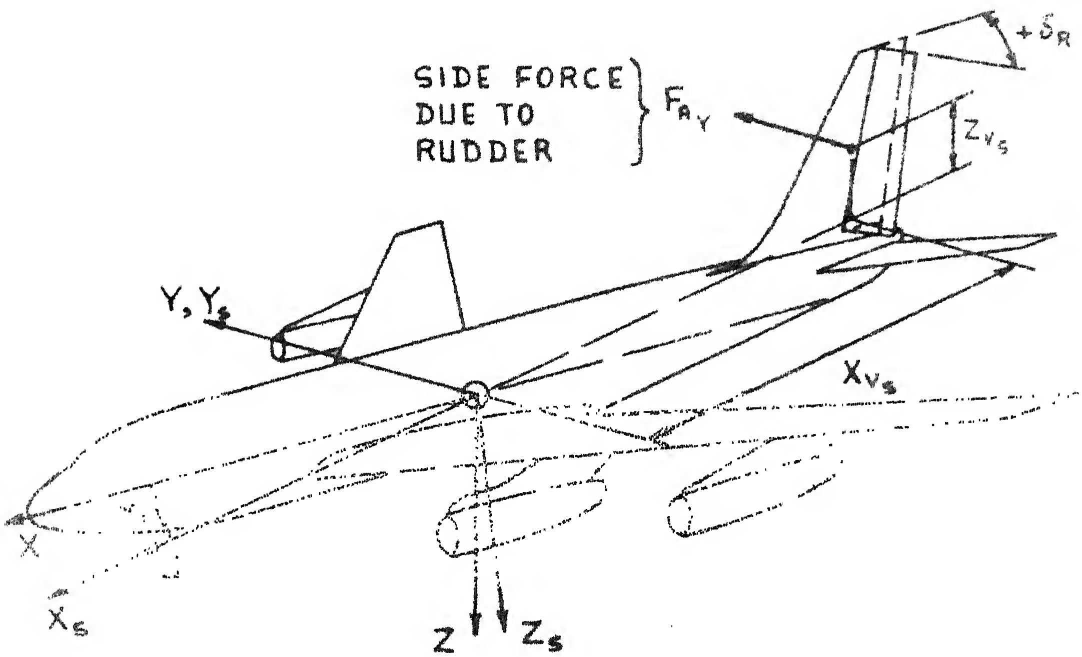


Figure B.3 Physical Explanation of Rolling Moment due to Rudder

APPENDIX C

LISTING OF MAIN PROGRAM

```
      DIMENSION A(25,25),B(25,10),C(10,25),AO(25,10)
      DIMENSION D(10,10),CI(25),R(10),ID(10)
      DIMENSION TEMP1(25,25),EM(25),EM1(10,25),EJ1(10,25)
      DIMENSION V(25,25)
      DIMENSION PC(25),PO(25),PE(25),POL(25)
      DIMENSION ATEMP(25,10),TEMP(25,25)
      DIMENSION AI(25,25),BI(25),RI(25,25),QIK(25),FK(25,25)
      DIMENSION G(10,10)
      DIMENSION AHAT(25,25),BHAT(25,10),CHAT(10,25)
      DIMENSION AC(25,25),BC(25,10),CC(10,25),Q(25,25),T2(25,25)
      DIMENSION DIOV(10,10),FL(10),FM(25)
      DIMENSION ASTAR(10,25)
      DIMENSION ABAR(25,25),BBAR(25,10),CBAR(10,25)
      DIMENSION ATIL(25,25),BTIL(25,10),CTIL(10,25),IROW(275),ICOL(25)
      DIMENSION H(275,25),ELL(25,25),T1(25,25),IP(12)
5     FORMAT(2I3,E8.1)
6     FORMAT(3H N=,I2,3X,2HM=,I2,3X,15HZERO TOLERANCE=,F8.1)
10    READ*, N,M,TOL
      IF(N.EQ.0)GO TO 999
      PRINT 6, N,M,TOL
15    FORMAT(6(F11.5,2X))
      DO 20 I=1,N
20    READ*,(A(I,J),J=1,N)
      DO 21 I=1,N
21    READ*,(B(I,J),J=1,M)
      DO 22 I=1,M
22    READ*,(C(I,J),J=1,N)
25    FORMAT(10X,8HA MATRIX)
      PRINT 25
      DO 26 I=1,N
26    PRINT 15, (A(I,J),J=1,N)
27    FORMAT(10X,8HB MATRIX)
      PRINT 27
      DO 28 I=1,N
28    PRINT 15, (B(I,J),J=1,M)
29    FORMAT(10X,8HC MATRIX)
      PRINT 29
      DO 30 I=1,M
30    PRINT 15,(C(I,J),J=1,N)
      I=1
      CALL ARRAY (2,N,N,25,25,A,A)
      CALL ARRAY (2,N,M,25,10,B,B)
      CALL ARRAY (2,M,N,10,25,C,C)
35    J=0
      CALL MCPY (B,AO,N,M,0)
      CALL RCPY (C,1,CI,M,N,0)
38    CALL GAPRD (CI,AO,R,1,N,M)
39    K=0
40    K=K+1
      IF (R(K)-TOL) 41,41,43
41    IF (R(K)+TOL) 43,42,42
42    IF (M-K) 48,50,40
43    ID(I)=J
      DO 44 K=1,M
44    D(I,K)=R(K)
      IF (M-I) 48,60,45
45    I=I+1
```

```
      GO TO 35
48    PRINT 49
49    FORMAT (14H PROGRAM ERROR)
      GO TO 10
50    CALL GMPRD (A,AO,ATEMP,N,N,N)
      CALL MCPY (ATEMP,AO,N,N,0)
      J=J+1
      CALL GMPRD (CI,AO,P,1,N,N)
      IF (N-1-J) 48,43,39
57    PRINT 58
58    FORMAT (21H D MATRIX IS SINGULAR)
      JOTEST=1
      GO TO 65
60    CALL ARRAY (2,M,M,10,10,D,D)
      CALL MCPY (D,DINV,M,M,0)
      CALL MINV (DINV,M,DET,EL,R)
      JOTEST=0
      IF (DET-TOL) 61,61,65
61    IF (DET+TOL) 65,62,62
62    GO TO 57
65    PRINT 67,DET
67    FORMAT (25H DETERMINANT OF D MATRIX=E10.3)
      CALL ARRAY (1,M,M,10,10,D,D)
      CALL ARRAY (1,M,M,10,10,DINV,DINV)
68    FORMAT (10X,8HD MATRIX)
      PRINT 68
      DO 968 I=1,M
968   PRINT 15,(D(I,J),J=1,M)
69    FORMAT (10X,16HD INVERSE MATRIX)
      PRINT 69
      DO 969 I=1,M
969   PRINT 15,(DINV(I,J),J=1,M)
971   FORMAT (7H D-SUB-,I2,1H=,I2)
      DO 972 K=1,M
972   PRINT 971,K,ID(K)
      L=1
      IF(JOTEST) 48,70,550
70    I=1
      CALL RCPY (C,I,CI,M,N,0)
71    CALL GMPRD (CI,A,EN,1,N,N)
      CALL MCPY (EN,CI,1,N,0)
      J=J+1
      IF (J-ID(L)-1) 71,72,48
72    DO 73 K=1,N
73    ASTAR(I,K)=CI(K)
74    IF(I-M) 75,980,48
75    I=I+1
      GO TO 70
80    FORMAT (10X,13HA STAR MATRIX)
980   PRINT 80
      DO 981 I=1,M
981   PRINT 15,(ASTAR(I,J),J=1,N)
      CALL ARRAY (2,M,N,10,25,ASTAR,ASTAR)
      CALL ARRAY (2,M,M,10,10,DINV,DINV)
      CALL GMPRD (B,DINV,BBAR,N,M,M)
      CALL GMPRD (BBAR,ASTAR,TEMP,N,M,N)
      CALL GMSUB (A,TEMP,ABAR,N,N)
```

```
      CALL MCPY (C,CBAR,M,N,0)
      CALL GMPRD(DINV,ASTAR,TEMP,M,M,N)
      CALL ARRAY (1,M,N,25,25,TEMP,TEMP)
      CALL ARRAY (1,N,N,25,25,ABAR,ABAR)
      CALL ARRAY (1,N,M,25,10,BBAR,BBAR)
      CALL ARRAY (1,M,N,10,25,CBAR,CBAR)
81    FORMAT (10X,12HA BAR MATRIX)
      PRINT 81
      DO 982 I=1,N
982   PRINT 15,(ABAR(I,J),J=1,N)
82    FORMAT (10X,12HB BAR MATRIX)
      PRINT 82
      DO 983 I=1,N
983   PRINT 15,(BBAR(I,J),J=1,M)
83    FORMAT (10X,12HC BAR MATRIX)
      PRINT 83
      DO 84 I=1,M
84    PRINT 15,(CBAR(I,J),J=1,N)
85    FORMAT (10X,22HD-INVERSE*ASTAR MATRIX)
      PRINT 85
      DO 86 I=1,M
86    PRINT 15,(TEMP(I,J),J=1,N)
C
C CALCULATE T1 MATRIX
C
      K=0
      CALL ARRAY (2,N,M,25,10,BBAR,BBAR)
      CALL ARRAY (2,N,N,25,25,ABAR,ABAR)
      CALL MCPY (BBAR,AO,N,M,0)
      CALL SCLA (H,0,N*M,N,0)
100   CALL GMTRA (AO,TEMP,N,M)
      DO 101 JJ=1,M
      J=JJ
101   CALL RADD (TEMP,J,H,K+J,M,N,0,N*M)
      IF (K-(N-1)*M) 108,109,48
108   K=K+M
      CALL MCPY (AO,ATEMP,N,M,0)
      CALL GMPRD (ABAR,ATEMP,AO,N,N,M)
      GO TO 100
109   CALL MFGR (H,N*M,N,TOL,NC,IROW,ICOL)
      IF (NC-N) 122,120,48
120   IP(M+2)=0
      CALL MCPY (ABAR,ATIL,N,N,0)
      CALL MCPY (BBAR,BTIL,N,M,0)
      CALL ARRAY (2,M,N,10,25,CBAR,CBAR)
      CALL MCPY (CBAR,CTIL,M,N,0)
      CALL SCLA (T1,0,N,N,0)
      CALL DCLA (T1,1.0,N,0)
      CALL MCPY (T1,ELL,N,N,0)
121   FORMAT (35H SYSTEM IS CONTROLLABLE T1 MATRIX=I)
      PRINT 121
      GO TO 199
122   PRINT 123,NC
123   FORMAT (31H SYSTEM IS NOT CONTROLLABLE NC=,I2)
      CALL SCLA (ELL,0,N,N,0)
      CALL ARRAY (1,N*M,N,275,25,H,H)
      PRINT 125
```

```
      CALL MCPY (C,CBAR,M,N,0)
      CALL GMPRD(DINV,ASTAR,TEMP,M,M,N)
      CALL ARRAY (1,M,N,25,25,TEMP,TEMP)
      CALL ARRAY (1,N,N,25,25,ABAR,ABAR)
      CALL ARRAY (1,N,M,25,10,BBAR,BBAR)
      CALL ARRAY (1,M,N,10,25,CBAR,CBAR)
81    FORMAT (10X,12HA BAR MATRIX)
      PRINT 81
      DO 982 I=1,N
982   PRINT 15,(ABAR(I,J),J=1,N)
82    FORMAT (10X,12HB BAR MATRIX)
      PRINT 82
      DO 983 I=1,N
983   PRINT 15,(BBAR(I,J),J=1,M)
83    FORMAT (10X,12HC BAR MATRIX)
      PRINT 83
      DO 84 I=1,M
84    PRINT 15,(CBAR(I,J),J=1,N)
85    FORMAT (10X,22HD-INVERSE*ASTAR MATRIX)
      PRINT 85
      DO 86 I=1,M
86    PRINT 15,(TEMP(I,J),J=1,N)
C
C CALCULATE T1 MATRIX
C
      K=0
      CALL ARRAY (2,N,M,25,10,BBAR,BBAR)
      CALL ARRAY (2,N,N,25,25,ABAR,ABAR)
      CALL MCPY (BBAR,AO,N,M,0)
      CALL SCLA (H,0,N*M,N,0)
100   CALL GMTRA (AO,TEMP,N,M)
      DO 101 JJ=1,M
      J=JJ
101   CALL RADD (TEMP,J,H,K+J,M,N,0,N*M)
      IF (K-(N-1)*M) 108,109,48
108   K=K+M
      CALL MCPY (AO,ATEMP,N,M,0)
      CALL GMPRD (ABAR,ATEMP,AO,N,N,M)
      GO TO 100
109   CALL MFGR (H,N*M,N,TOL,NC,IROW,ICOL)
      IF (NC-N) 122,120,48
120   IP(M+2)=0
      CALL MCPY (ABAR,ATIL,N,N,0)
      CALL MCPY (BBAR,BTIL,N,M,0)
      CALL ARRAY (2,M,N,10,25,CBAR,CBAR)
      CALL MCPY (CBAR,CTIL,M,N,0)
      CALL SCLA (T1,0,N,N,0)
      CALL DCLA (T1,1,0,N,0)
      CALL MCPY (T1,ELL,N,N,0)
121   FORMAT (35H SYSTEM IS CONTROLLABLE T1 MATRIX=I)
      PRINT 121
      GO TO 199
122   PRINT 123,NC
123   FORMAT (31H SYSTEM IS NOT CONTROLLABLE NC=,I2)
      CALL SCLA (ELL,0,N,N,0)
      CALL ARRAY (1,N*M,N,275,25,H,H)
      PRINT 125
```

```
158     FORMAT (10X,14HA TILDA MATRIX)
        DO 159 I=1,N
159     PRINT 15,(ATIL(I,J),J=1,N)
        PRINT 160
160     FORMAT (10X,14HB TILDA MATRIX)
        DO 161 I=1,N
161     PRINT 15,(BTIL(I,J),J=1,M)
        PRINT 162
162     FORMAT (10X,14HC TILDA MATRIX)
        DO 163 I=1,M
163     PRINT 15,(CTIL(I,J),J=1,N)
        CALL ARRAY (2,N,N,25,25,ATIL,ATIL)
        CALL ARRAY (2,N,M,25,10,BTIL,BTIL)
        CALL ARRAY (2,M,N,10,25,CTIL,CTIL)
        CALL ARRAY (2,N,N,25,25,T1,T1)
C
C       CALCULATE Q MATRIX
C
199     I=0
        IPE=0
        ISUM=0
        CALL XCPY (ATIL,AC,1,1,NC,NC,N,N,0)
        CALL XCPY (BTIL,BC,1,1,NC,M,N,M,0)
        CALL XCPY (CTIL,CC,1,1,M,NC,M,N,0)
        CALL SCLA (Q,0,N,N,0)
        DO 200  K=1,M
200     ISUM=ID(K)+1+ISUM
        IF (ISUM-NC) 201,250,9201
9201    PRINT 9202
9202    FORMAT (48H THE SUM (D-SUB-I+1)FOR I=1,M IS GREATER THAN NC)
        GO TO 48
201     I=I+1
202     CALL SCLA (H,0,NC*M+ID(I)+1,NC,0)
        K=0
        CALL MCPY (BC,A0,NC,M,0)
210     CALL GMTRA (A0,TEMP,NC,M)
        DO 211  JJ=1,M
        J=JJ
211     CALL RADD (TEMP,J,H,K+J,M,NC,0,NC*M+ID(I)+1)
        K=K+M
        IF (K-NC*M) 212,213,48
212     CALL MCPY (A0,ATEMP,N,M,0)
        CALL GMPRD (AC,ATEMP,A0,NC,NC,M)
        GO TO 210
213     NN=(NC-1)*M
        DO 214 L=0,NN,M
214     CALL SRMA (H,0,NC*M+ID(I)+1,NC,I+L,0)
217     CALL XCPY (CTIL,CC,1,1,M,NC,M,N,0)
        CALL RCPY (CC,I,CI,M,NC,0)
        II=ID(I)+1
        DO 219 J=1,II
        CALL RADD (CI,1,H,NC*M+J,1,NC,0,NC*M+ID(I)+1)
        CALL RADD (CI,1,Q,IPE+J,1,NC,0,NC)
        CALL MCPY (CI,EN,1,NC,0)
219     CALL GMPRD (EN,AC,CI,1,NC,NC)
220     CALL MFGR (H,NC*M+ID(I)+1,NC,TOL,JR,IROW,ICOL)
        IP(I)=NC-JR+ID(I)+1
```

```
      PRINT 9220,I,IP(I)
9220  FORMAT (7H P-SUB-,I2,1H=,3X,I2)
      IPE=IPE+IP(I)
      IF (IP(I)-ID(I)-1) 48,230,221
221   CALL ARRAY (1,NC*M+ID(I)+1,NC,275,25,H,H)
      PRINT 9221,I
9221  FORMAT (25H SOLUTION OF ETA*H FOR I=,I2)
      INC=NC*M+ID(I)+1
      DO 9222 L=1,INC
9222  PRINT 15,(H(L,L1),L1=1,NC)
      PRINT 9127,(ICOL(L),L=1,N)
      NNN=NC-JR
      DO 223 J=1,NNN
      CALL SCLA (EN,0,1,NC,0)
      EN(ICOL(JR+J))=1
      DO 222 L=1,JR
222   EN(ICOL(L))=H(L,JR+J)
223   CALL RADD (EN,1,0,ID(I)+1+J+IPE-IP(I),1,NC,0,NC)
230   IF (I-M) 231,270,48
231   ISUM=0
      III=I+1
      DO 232 J=III,M
232   ISUM=ISUM+ID(J)+1
      IF (IPE+ISUM-NC) 201,250,48
250   I=I+1
      IP(I)=ID(I)+1
      PRINT 251,I,IP(I)
251   FORMAT (7H P-SUB-,I2,1H=,I2)
      CALL RCPY (CC,1,CI,M,NC,0)
      II=ID(I)+1
      DO 252 L=1,II
      CALL RADD (CI,1,0,IPE+L,1,NC,0,NC)
      CALL MCPY (CI,EN,1,NC,0)
252   CALL GMPRO (EN,AC,CI,1,NC,NC)
      IPE=IPE+IP(I)
      IF (I-M) 250,270,48
270   IF (IPE-NC) 271,275,48
271   CALL MCPY (Q,TEMP,NC,NC,0)
      CALL MFGR (TEMP,NC,NC,TOL,JRANK,IROW,ICOL)
      IF(JRANK-IPE) 48,272,48
272   CALL ARRAY (1,NC,NC,25,25,TEMP,TEMP)
      PRINT 9272
9272  FORMAT (16H SOLUTION OF Q*X)
      DO 9273 L=1,NC
9273  PRINT 15,(TEMP(L,L1),L1=1,NC)
      PRINT 9127,(ICOL(L),L=1,N)
      NNNN=NC-IPE
      DO 274 J=1,NNNN
      CALL SCLA (EN,0,1,NC,0)
      EN(ICOL(IPE+J))=1
      DO 273 L=1,IPE
273   EN(ICOL(L))=TEMP(L,IPE+J)
274   CALL RADD (EN,1,0,IPE+J,1,NC,0,NC)
      IP(M+1)=NC-IPE
      MMM=M+1
      PRINT 9274,MMM,IP(M+1)
9274  FORMAT (7H P-SUB-,I2,1H=,I2)
```

```
275       IF (NC-N) 276,280,48
276       CALL SCLA (TEMP,0,N,N,0)
          NCC1=NC+1
          CALL ARRAY (1,N,N,25,25,TEMP,TEMP)
          DO 9276 J=NCC1,N
9276      TEMP(J,J)=1.0
          CALL ARRAY (2,N,N,25,25,TEMP,TEMP)
          DO 277 JJ=1,NC
          CALL SCLA (EN,0,1,N,0)
          J=JJ
          CALL RCPY (Q,J,EN,NC,NC,0)
277       CALL RADD (EN,1,TEMP,J,1,N,0,N)
          GO TO 281
280       CALL MCPY (Q,TEMP,N,N,0)
281       CALL MCPY (TEMP,T2,N,N,0)
          CALL MINV (TEMP,N,DETER,CI,EN)
          CALL ARRAY (1,N,N,25,25,T2,T2)
          PRINT 282
282       FORMAT (10X,9HT2 MATRIX)
          DO 283 I=1,N
283       PRINT 15,(T2(I,J),J=1,N)
          IF(DETER-TOL) 284,284,287
284       IF(DETER+TOL) 287,285,285
285       PRINT 286
286       FORMAT (22H T2 MATRIX IS SINGULAR)
          GO TO 48
287       CALL ARRAY (2,N,N,25,25,T2,T2)
          CALL GMPRD (CTIL,TEMP,CHAT,M,N,N)
          CALL GMPRD (T2,BTIL,BHAT,N,N,M)
          CALL GMPRD (T2,ATIL,AHAT,N,N,N)
          CALL MCPY (AHAT,ELL,N,N,0)
          CALL GMPRD (ELL,TEMP,AHAT,N,N,N)
          CALL ARRAY (1,N,N,25,25,TEMP,TEMP)
          CALL ARRAY (1,N,N,25,25,AHAT,AHAT)
          CALL ARRAY (1,N,M,25,10,BHAT,BHAT)
          CALL ARRAY (1,M,N,10,25,CHAT,CHAT)
          PRINT 288
288       FORMAT (10X,17HT2 INVERSE MATRIX)
          DO 289 I=1,N
289       PRINT 15,(TEMP(I,J),J=1,N)
          PRINT 290
290       FORMAT (10X,12HA-HAT MATRIX)
          DO 291 I=1,N
291       PRINT 15,(AHAT(I,J),J=1,N)
          PRINT 292
292       FORMAT (10X,12HB-HAT MATRIX)
          DO 293 I=1,N
293       PRINT 15,(BHAT(I,J),J=1,M)
          PRINT 294
294       FORMAT (10X,12HC-HAT MATRIX)
          DO 295 I=1,M
295       PRINT 15,(CHAT(I,J),J=1,N)
C
C CALCULATE THE G SUB I MATRICES
C
400       DO 406 II=1,M
          CALL SCLA (G,0,M,M,0)
```

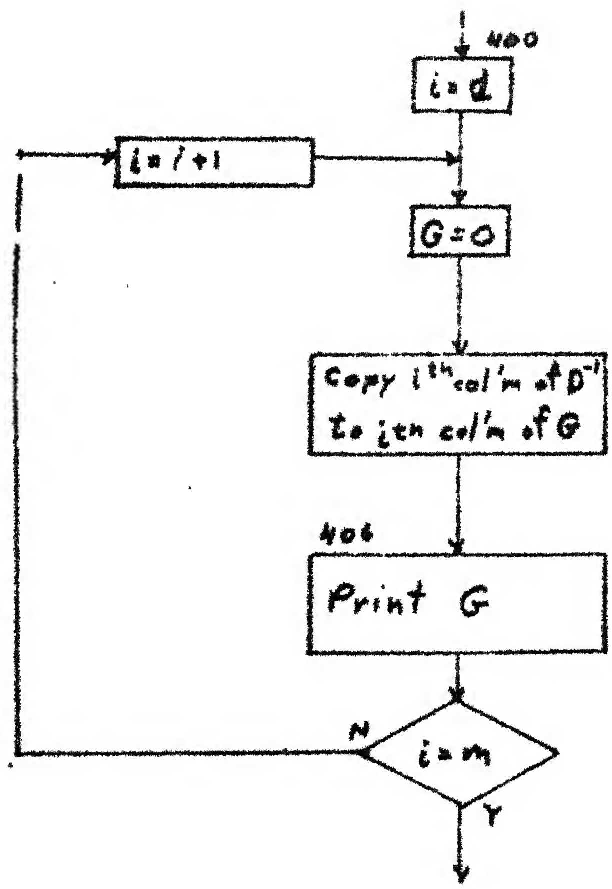
```
      I=II
      CALL CAOD (OINV,I,G,I,M,M,0,M)
      PRINT 405,I
405   FORMAT (6H G-SUB,I2,2X,6HMATRIX)
      CALL ARRAY (1,M,M,10,10,G,G)
      DO 406 J=1,M
406   PRINT 15,(G(J,K),K=1,M)
C
C     CALUCULATE THE PI I K CONSTANTS AND J MATRICES
C
      CALL ARRAY (2,N,N,25,25,AHAT,AHAT)
      CALL ARRAY (2,N,M,25,10,BHAT,BHAT)
500   IPE=0
      I=1
501   CALL XCPY (AHAT,AI,IPE+1,IPE+1,IP(I),IP(I),N,N,0)
      CALL XCPY (BHAT,BI,IPE+1,I,IP(I),1,N,M,0)
      CALL SCLA (RI,0,IP(I),IP(I),0)
      CALL DCLA (RI,1,0,IP(I),0)
      CALL DCPY (AI,EN,IP(I),0)
      SUM=0
      IPI=IP(1)
      DO 502 K=1,IPI
502   SUM=SUM+EN(K)
      QIK(1)=SUM
      J=0
      CALL SCLA (FK,0,N,N,0)
503   CALL GMPRD (PI,BI,EN,IP(I),IP(I),1)
      CALL CADD (EN,1,FK,IP(I)-J,IP(I),1,0,IP(I))
      J=J+1
      CALL SCLA (TEMP1,0,N,N,0)
      CALL DCLA (TEMP1,QIK(J),IP(I),0)
      CALL GMPRD(RI,AI,Q,IP(I),IP(I),IP(I))
      CALL GMSUB(Q,TEMP1,RI,IP(I),IP(I))
      IF (J-IP(I)) 504,510,48
504   CALL GMPRD(RI,AI,Q,IP(I),IP(I),IP(I))
      CALL DCPY (Q,EN,IP(I),0)
      SUM=0
      DO 505 K=1,IPI
505   SUM=SUM+EN(K)
      QIK(J+1)=SUM/(J+1)
      GO TO 503
510   CALL ARRAY (1,IP(I),IP(I),25,25,RI,RI)
      DO 9513 K=1,IPI
      DO 9513 L=1,IPI
512   SUM=RI(K,L)
      IF (SUM-TOL) 513,513,514
513   IF (SUM+TOL) 514,9513,9513
9513  CONTINUE
9514  GO TO 520
514   PRINT 515,J
515   FORMAT (26H TRACE METHOD FAILS FOR I=,I2)
520   PRINT 521,I,IP(I)
521   FORMAT (3H I=,I2,3X,8HP-SUB-I=,I2)
      DO 523 K=1,IPI
523   PRINT 524,I,K,QIK(K)
524   FORMAT (2H Q,I2,I2,1H=,E10.3)
      GO TO 600
```

```
9525   CONTINUE
       CALL MCPY (FK,V,IP(I),IP(I),0)
       CALL ARRAY (1,IP(I),IP(I),25,25,FK,FK)
       PRINT 9524
9524   FORMAT (10X,15HTHE K MATRIX IS)
       DO 525 K=1,IPI
525    PRINT 15,(FK(K,L),L=1,IPI)
       CALL MINV (V,IP(I),DET,EN,EM)
       IF (DET-TOL) 526,526,530
526    IF (DET+TOL) 530,527,527
527    PRINT 528
528    FORMAT (21H K MATRIX IS SINGULAR)
       GO TO 48
530    K=0
9530   FORMAT (10X,15HTHE V MATRIX IS)
       PRINT 9530
       CALL ARRAY (1,IP(I),IP(I),25,25,V,V)
       DO 9532 L=1,IPI
9532   PRINT 15,(V(L,L1),L1=1,IPI)
       CALL ARRAY (2,IP(I),IP(I),25,25,V,V)
9531   K=K+1
       CALL SCLA (EM1,0,M,N,0)
       CALL RCPY (V,K,EN,IP(I),IP(I),0)
       CALL SCLA (EM,0,1,25,0)
       DO 531 L=1,IPI
531    EM(IPE+L)=EN(L)
       CALL RADD (EM,1,EM1,I,1,N,0,M)
       CALL GMPRD (DINV,EM1,EJ1,M,M,N)
       CALL GMPRD (EJ1,T2,TEMP1,M,N,N)
       CALL GMPRD (TEMP1,T1,EJ1,M,N,N)
       CALL ARRAY (1,M,N,10,25,EJ1,EJ1)
       CALL ARRAY (1,M,N,10,25,EM1,EM1)
       PRINT 535,I
       PRINT 532
532    FORMAT (15H MATRIX    M    =)
       KKKK=IP(I)+1-K
       PRINT 535,KKKK
       DO 533 L=1,M
533    PRINT 15,(EM1(L,L1),L1=1,N)
535    FORMAT (10X,I2)
       PRINT 535,I
536    FORMAT (15H MATRIX    J    =)
       PRINT 536
       KKKK=IP(I)+1-K
       PRINT 535,KKKK
       DO 537 L=1,M
537    PRINT 15,(EJ1(L,L1),L1=1,N)
       IF (K-IP(I)) 9531,540,48
540    IF (I-M) 541,550,48
541    IPE=IPE+IP(I)

       I=I+1
       GO TO 501
550    IPE=IPE+IP(I)
       IF (IPE-NC) 560,10,48
560    I=I+1
       GO TO 501
```

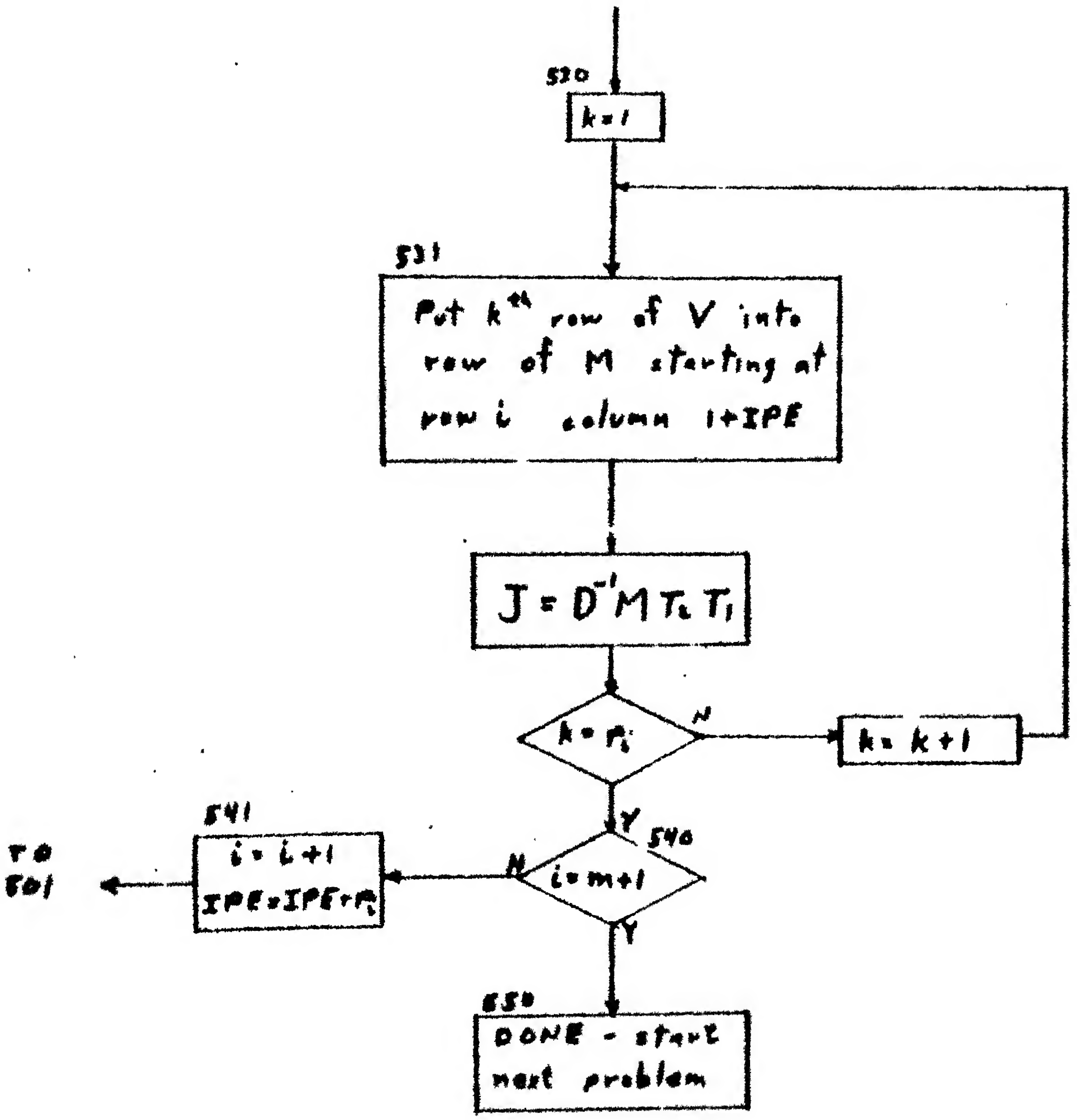
```
600     IPI=IP(I)
        IF (I-M) 627,627,626
626     IRI=IPI
        GO TO 620
627     IRI=IP(I)-ID(I)-1
620     IF (IRI) 18,623,621
621     IF(ABS(QIK(IRI))-TOL) 622,622,625
622     IRI=IRI-1
        GO TO 620
623     IF (I-M) 9525,9525,10
625     PC(IRI+1)=1.0
        DO 601 IA=1,IRI
601     PC(IA)=-QIK(IRI+1-IA)
        CALL PROD(PC,IRI+1,PO,PE,POL,IR,IER)
603     FORMAT (22H SUBROUTINE PROD FAILS)
604     PRINT 605,I
605     FORMAT(41H NON-ZERO EIGENVALUES OF MATRIX AHAT-SUB-,I2)
        IF (IER) 606,606,610
606     PRINT 9606
9606    FORMAT (10H REAL PART,3X,14HIMAGINARY PART)
        DO 607 IA=1,IR
607     PRINT 608,PO(IA),PE(IA)
608     FORMAT (E10.3,3X,E10.3)
        IF(I-M)9525,9525,10
610     PRINT 603
        IF (I-M) 9525,9525,10
999     END
```

Calculate the G_i matrices



Note : The numbers on the flowchart correspond to statement numbers in the program listing.

Calculate J_k^i



SUBROUTINES PACKAGE

The names and functions of the subroutines called by the main program listed above are appended below.

1. ARRAY : Convert data ARRAY from single to double dimension or vice versa. This subroutine is used to link the user program which has double dimension arrays and the SSP subroutines which operate on arrays of data in a vector fashion.
2. MCPY : Copy entire matrix.
3. RCPY : Copy specified row of a matrix into a vector.
4. GMPRD : Multiply two general matrices.
5. MINV : Invert a matrix.
6. SCLA : Set each element of a matrix equal to a given scalar.
7. GMTRA : Transpose a general matrix.
8. RADD : Add row of one matrix to row of another matrix.
9. MFGR : For a given matrix(M by N) E (a) determine rank and linearly independent rows and columns (basis) (b) find an orthogonal basis for the null space of E.
10. DCLA : Set each diagonal element of a matrix equal to a scalar.
11. XCPY : Copy a portion of a matrix.
12. SRMA : Multiply row of a matrix by a scalar and add to another row of the same matrix.
13. GMSUB : Subtract one general matrix from another to form resultant matrix.
14. CADD : Add column of one matrix to column of another matrix.
15. DCPY : Copy diagonal elements of a matrix into a vector.
16. PRQD : Calculate all real and complex roots of a given polynomial with real coefficient.

MODIFICATIONS IN THE PROGRAM

1. All the DIMENSION statements were brought on the top of the listing as per the requirement of DEC-1090.

2. READ statements were made format-free for the sake of convenience and to avoid confusion.

3. Device numbers (Data Sets) from the READ and WRITE statements were removed.

4. Since, the program itself did not have any halt statement, an additional statement

   ```
   IF (N.EQ.0)  GO TO 999
   ```

   was inserted. Execution is terminated when the Input file reads 0 for N.

5. Hollerith field specification was modified in certain format statements for printing of headings and variable names.

6. Do loop index modifications (to differentiate from the subroutine parameters) were performed to remove warnings.

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